3.1 CODES

The postulates of a mathematical system forms the basic assumption from which it is possible to deduce the theorems, laws and properties of the system.

The most common postulates used to formulate various structures are

i) **Closure**:

A set S is closed w.r.t. a binary operator, if for every pair of elements of S, the binary operator specifies a rule for obtaining a unique element of S.

The result of each operation with operator (+) or (.) is either 1 or 0 and 1, 0 ЄB.

ii) **Identity element**:

A set S is said to have an identity element w.r.t a binary operation * on S, if there exists an element e Є S with the property,

e* x = x * e = x

Eg: 0+ 0 = 0 0+ 1 = 1+ 0 = 1 a) **x+ 0= x**

1 . 1 = 1 1 . 0 = 0 . 1 = 1 b) **x. 1 = x**

iii) **Commutative law**:

binary operator * on a set S is said to be commutative if,

$\boxed{x * y = y * x}$ for all x, y Є S

Eg: 0+ 1 = 1+ 0 = 1 a) **x+ y= y+ x**

0 . 1 = 1 . 0 = 0 b) **x. y= y. x**

If * and • are two binary operation on a set S, • is said to be distributive over **+** whenever,

x . (y+ z) = (x. y) + (x. z)

Similarly, **+** is said to be distributive over • whenever,

x + (y. z) = (x+ y). (x+ z)

iv) **Inverse**:

A set S having the identity element e, w.r.t. binary operator * is said to have an inverse, whenever for every x Є S, there exists an element x' Є S such that,

x. **x'** Є e

a) x+ x' = 1, since 0 + 0' = 0+ 1 and 1+ 1' = 1+ 0 = 1

b) x. x' = 1, since 0 . 0' = 0. 1 and 1. 1' = 1. 0 = 0

Summary:

Table: 1.1 - Postulates of Boolean algebra:

POSTULATES	(a)	(b)
Postulate 2 (Identity)	**x + 0 = x**	**x . 1 = x**
Postulate 3 (Commutative)	**x+ y = y+ x**	**x . y = y. x**
Postulate 4 (Distributive)	**x (y+ z) = xy+ xz**	**x+ yz = (x+ y). (x+ z)**
Postulate 5 (Inverse)	**x+x' = 1**	**x. x' = 0**

The theorems, like the postulates are listed in pairs; each relation is the dual of the one paired with it. The postulates are basic axioms of the algebraic structure and need no proof. The theorems must be proven from the postulates. The proofs of the theorems with one variable are presented below. At the right is listed the number of the postulate that justifies each step of the proof.

1) a) $x + x = x$

$x + x = (x + x) \cdot 1$ ---------------------------------- by postulate 2(b) [$x \cdot 1 = x$]

$= (x + x) \cdot (x + x')$ ------------------- 5(a) [$x + x' = 1$]

$= x + xx'$ ------------------- 4(b) [$x + yz = (x+y)(x+z)$]

$= x + 0$ ------------------- 5(b) [$x \cdot x' = 0$]

$= x$ ------------------- 2(a) [$x + 0 = x$]

b) $x \cdot x = x$

$x \cdot x = (x \cdot x) + 0$ ------------------- by postulate 2(a)

[$x + 0 = x$]

$= (x \cdot x) + (x \cdot x')$ ------------------- 5(b) [$x \cdot x' = 0$]

$= x(x + x')$ ------------------- 4(a) [$x(y+z) = (xy) + (xz)$]

$= x(1)$ ------------------- 5(a) [$x + x' = 1$]

$= x$ ------------------- 2(b) [$x \cdot 1 = x$]

2) a) $x + 1 = 1$

$x + 1 = 1 \cdot (x + 1)$ ------------------- by postulate 2(b) [$x \cdot 1 = x$]

$= (x + x') \cdot (x + 1)$ ------------------- 5(a) [$x + x' = 1$]

$= x + x' \cdot 1$ ------------------- 4(b) [$x + yz = (x+y)(x+z)$]

$= x + x'$ ------------------- 2(b) [$x \cdot 1 = x$]

$= 1$ ------------------- 5(a) [$x + x' = 1$]

b) **$x \cdot 0 = 0$**

3) $(x')' = x$

From postulate 5, we have $x + x' = 1$ and $x \cdot x' = 0$, which defines the complement of x. The complement of x' is x and is also $(x')'$. Therefore, since the complement is unique,

$(x')' = x$.

4) Absorption Theorem:

a) $x + xy = x$

$x + xy = x.1 + xy$ ------------------- by postulate 2(b) [$x.1 = x$]

$= x(1 + y)$ ------------------- 4(a) [$x(y+z) = (xy) + (xz)$]

$= x(1)$ ------------------- by theorem 2(a) [$x + 1 = x$]

$= x.$ ------------------- by postulate 2(a) [$x.1 = x$]

b) $x.(x + y) = x$

$x.(x + y) = x.x + x.y$ ------------------- 4(a) [$x(y+z) = (xy) + (xz)$]

$= x + x.y$ ------------------- by theorem 1(b) [$x.x = x$]

$= x.$ ------------------- by theorem 4(a) [$x + xy = x$]

c) $x + x'y = x + y$ $x + x'y = x + xy + x'y$ -------------------by theorem 4(a) [$x + xy = x$]

$= x + y(x + x')$ -------------------------- by postulate 4(a) [$x(y+z) = (xy) + (xz)$]

$= x + y(1)$ ------------------- 5(a) [$x + x' = 1$]

$= x + y$ ------------------- 2(b) [$x.1 = x$]

d) $x.(x' + y) = xy$

$x.(x' + y) = x.x' + xy$--------------------------- by postulate 4(a) [$x(y+z) = (xy) + (xz)$]

$= 0 + xy$ ------------------- 5(b) [$x.x' = 0$]

$= xy.$ ------------------- 2(a) [$x + 0 = x$]

Properties of Boolean algebra:

1. Commutative property:

Boolean addition is commutative, given by

x+ y = y+ x

According to this property, the order of the OR operation conducted on the variables makes no difference.

Boolean algebra is also commutative over multiplication given by,

x. y = y. x

This means that the order of the AND operation conducted on the variables makes no difference.

2. Associative property:

The associative property of addition is given by,

$\boxed{\textbf{A+ (B+ C) = (A+B) + C}}$

The OR operation of several variables results in the same, regardless of the grouping of the variables.

The associative law of multiplication is given by,

A. (B. C) = (A.B) . C

It makes no difference in what order the variables are grouped during the AND operation of several variables.

3. Distributive property:

The Boolean addition is distributive over Boolean multiplication, given by

A+ BC = (A+B) (A+C)

The Boolean addition is distributive over Boolean addition, given by

A. (B+C) = (A.B)+ (A.C)

4. Duality:

It states that every algebraic expression deducible from the postulates of Boolean algebra remains valid if the operators and identity elements are interchanged.

If the dual of an algebraic expression is desired, we simply interchange OR and AND operators and replace 1's by 0's and 0's by 1's.

$x + x' = 1$ is **$x . x' = 0$**

Duality is a very important property of Boolean algebra.

Summary:

Table: 1.2 - Theorems of Boolean algebra:

	THEOREMS	(a)	(b)
1	Idempotent	$x + x = x$	$x . x = x$
		$x + 1 = 1$	$x . 0 = 0$
2	Involution	$(x')' = x$	
3	Absorption	$x + xy = x$	$x (x + y) = x$
		$x + x'y = x + y$	$x . (x' + y) = xy$
4	Associative	$x + (y + z) = (x + y) + z$	$x (yz) = (xy) z$
5	DeMorgan's Theorem	$(x + y)' = x' . y'$	$(x . y)' = x' + y'$

DeMorgan's Theorems:

Two theorems that are an important part of Boolean algebra were proposed by DeMorgan.

The first theorem states that the complement of a product is equal to the sum of the complements.

(AB)’ = A’+ B’

The second theorem states that the complement of a sum is equal to the product of the complements.

(A+ B)’ = A’. B’

Consensus Theorem:

In simplification of Boolean expression, an expression of the form **AB+ A’C+ BC**, the term BC is redundant and can be eliminated to form the equivalent expression AB+
A’C. The theorem used for this simplification is known as consensus theorem and is stated as,

AB+ A’C+ BC = AB+ A’C

The dual form of consensus theorem is stated as,

(A+B) (A’+C) (B+C) = (A+B) (A’+C)