5.2 Finding Inverse Z-transform by Residue Method

Procedure:

1. write $F(z)$ from given expression and write $F(z)z^{n-1}$
2. Find the poles by equating denominator to zero in $F(z)z^{n-1}$
3. Write the order of poles
4. Find the residue at these poles

Case i: If $z=a$ is pole of order 1 (or) simple pole then

$$\left[\operatorname{Res} F(z)z^{n-1}\right]_{z=a} = \lim_{z\to a}(z-a)F(z)z^{n-1}$$

1 **Find** $Z^{-1}\left[\dfrac{2z}{(z-2)(z^2+1)}\right]$ **by the method of residues.**

Solution:

Let $F(z)=\dfrac{2z}{(z-1)(z^2+1)}$

$$F(z)z^{n-1}=\frac{2zz^{n-1}}{(z-1)(z^2+1)}$$

$$F(z)z^{n-1}=\frac{2z^n}{(z-1)(z+i)(z-i)} \quad -----(1)$$

Here $z=1$, $z=i$ and $z=-i$ are poles of order 1.

1) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=a} = \lim_{z\to a}(z-a)F(z)z^{n-1}$

$$\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=1} = \lim_{z\to 1}(z-1)\frac{2z^n}{(z-1)(z+i)(z-i)}$$

$$=\lim_{z\to 1}\frac{2z^n}{(z+i)(z-i)}$$

$$=\frac{2(1)^n}{(1+i)(1-i)}$$

$$=\frac{2}{2} \qquad \because (1+i)(1-i)=1^2-i^2=1-(-1)=1+1=2$$

$$\boxed{\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=1}=1}$$

2) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=i} = \lim_{z\to a}(z-i)F(z)z^{n-1}$

$$\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=i} = \lim_{z\to i}(z-i)\frac{2z^n}{(z-1)(z-i)(z+i)}$$

$$=\lim_{z\to i}\frac{2z^n}{(z-1)(z+i)}$$

$$=\frac{2(i)^n}{(i-1)(i+i)}$$

$$=\frac{2(i)^n}{2i(i-1)}$$

$$=\frac{(i)^n}{i(i-1)}=\frac{(i)^n}{(i^2-i)}=\frac{(i)^n}{(-1-i)}$$

$$\boxed{\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=i}=\frac{-(i)^n}{(1+i)}}$$

	3) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-i} = \lim_{z\to -i}(z+i)F(z)z^{n-1}$ $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-i} = \lim_{z\to -i}\cancel{(z+i)}\dfrac{2z^n}{(z-1)\cancel{(z+i)}(z-i)}$ $= \lim_{z\to -i}\dfrac{2z^n}{(z-1)(z-i)}$ $= \dfrac{2(-i)^n}{(-i-1)(-i-i)} = \dfrac{2(-i)^n}{(1+i)(2i)}$ $= \dfrac{(-i)^n}{(1+i)(i)} = \dfrac{(-i)^n}{(i+i^2)} = \dfrac{(-i)^n}{(i-1)}$ $\boxed{\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-i} = \dfrac{(-i)^n}{(i-1)}}$ $f(n) = \text{sum of residues of } F(z)z^{n-1}$ $\boxed{f(n) = 1 - \dfrac{(i)^n}{(1+i)} + \dfrac{(-i)^n}{(i-1)}}$
2.	**Find the inverse Z-Transform of** $\dfrac{z(z+1)}{(z-1)^3}$ **by residue method**. **Solution:** Let $F(z) = \dfrac{z(z+1)}{(z-1)^3}$ $F(z)z^{n-1} = \dfrac{zz^{n-1}(z+1)}{(z-1)^3}$ $F(z)z^{n-1} = \dfrac{z^n(z+1)}{(z-1)^3} \quad -----(1)$ $z=1$ is a pole of order 3 $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=a} = \dfrac{1}{(m-1)!}\lim_{z\to a}\dfrac{d^{m-1}}{dz^{m-1}}(z-a)^m F(z)z^{n-1}$ $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=1} = \dfrac{1}{(3-1)!}\lim_{z\to 1}\dfrac{d^2}{dz^2}\cancel{(z-1)^3}\dfrac{z^n(z+1)}{\cancel{(z-1)^3}}$ $= \dfrac{1}{2}\lim_{z\to 1}\dfrac{d^2}{dz^2}\left[z^{n+1}+z^n\right]$ $= \dfrac{1}{2}\lim_{z\to 1}\dfrac{d}{dz}\left[(n+1)z^n + nz^{n-1}\right]$ $= \dfrac{1}{2}\lim_{z\to 1}\left[(n+1)nz^{n-1} + n(n-1)z^{n-2}\right]$ $= \dfrac{1}{2}\lim_{z\to 1}\left[(n^2+n)(1)^{n-1} + (n^2-n)1^{n-2}\right]$ $= \dfrac{1}{2}\left[n^2+n+n^2-n\right]$ $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=1} = \dfrac{1}{2}\left[2n^2\right]$ $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=1} = n^2$

	$f(n) = \text{sum of residues of } F(z)z^{n-1} = n^2$
3.	**Find the inverse Z-transform of the function $\frac{z}{z^2+7z+10}$ by the method of residues.** **Solution:** $Z^{-1}\left[\frac{z}{z^2+7z+10}\right] = ?$ $F(z) = \frac{z}{z^2+7z+10} = \frac{z}{(z+2)(z+5)}$ $F(z)z^{n-1} = \frac{zz^{n-1}}{(z+2)(z+5)}$ $F(z)z^{n-1} = \frac{z^n}{(z+2)(z+5)} \quad ------(1)$ Here *z=-2* and *z=-5* are pole of order 1 1) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=a} = \lim_{z\to a}(z-a)F(z)z^{n-1}$ $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-2} = \lim_{z\to -2} \cancel{(z+2)} \frac{z^n}{\cancel{(z+2)}(z+5)}$ $= \frac{(-2)^n}{(-2+5)} = \frac{(-2)^n}{3}$ $\boxed{\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-2} = \frac{(-2)^n}{3}}$ 2) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-5} = \lim_{z\to -5} \cancel{(z+5)} \frac{z^n}{(z+2)\cancel{(z+5)}}$ $= \frac{(-5)^n}{(-5+2)} = \frac{(-5)^n}{-3}$ $\boxed{\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-5} = \frac{-(-5)^n}{3}}$ $f(n) = \text{sum of residues of } F(z)z^{n-1}$ $\boxed{f(n) = \frac{(-2)^n}{3} - \frac{(-5)^n}{3} = \frac{1}{3}\left[(-2)^n - (-5)^n\right]}$
4.	**Find $Z^{-1}\left[\frac{z^{-2}}{\left(1+z^{-1}\right)^2\left(1-z^{-1}\right)}\right]$ by using residue method.** **Solution:** $F(z) = \frac{z^{-2}}{\left(1+z^{-1}\right)^2\left(1-z^{-1}\right)} = \frac{1}{z^2\left(\frac{z+1}{z}\right)^2\left(\frac{z-1}{z}\right)}$ $F(z) = \frac{z}{\left(z+1\right)^2\left(z-1\right)}$ $F(z)z^{n-1} = \frac{zz^{n-1}}{\left(z+1\right)^2\left(z-1\right)}$

$$F(z)z^{n-1} = \frac{z^n}{(z+1)^2(z-1)} \quad -----(1)$$

Here $z=-1$ is pole of order 2, and $z=1$ is pole of order 1

1) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=a} = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m F(z)z^{n-1}$

$$\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-1} = \frac{1}{(2-1)!}\lim_{z\to -1}\frac{d^2}{dz^2}\cancel{(z+1)^2}\frac{z^n}{\cancel{(z+1)^2}(z-1)}$$

$$= \frac{1}{1!}\lim_{z\to -1}\frac{d}{dz}\left[\frac{z^n}{z-1}\right]$$

$$= \lim_{z\to -1}\left[\frac{(z-1)nz^{n-1} - z^n(1-0)}{(z-1)^2}\right]$$

$$= \frac{(-1-1)n(-1)^{n-1} - (-1)^n}{(-1-1)^2} = \frac{-2n(-1)^{n-1} - (-1)^n}{4} = \frac{(-1)^n}{4}[2n-1]$$

$$\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=-1} = \frac{(-1)^n}{4}[2n-1]$$

2) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=a} = \lim_{z\to a}(z-a)F(z)z^{n-1}$

$$\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=1} = \lim_{z\to 1}\cancel{(z-1)}\frac{z^n}{(z+1)^2\cancel{(z-1)}}$$

$$= \lim_{z\to 1}\frac{z^n}{(z+1)^2} = \frac{1^n}{(1+1)^2} = \frac{1}{2}$$

$$\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=1} = \frac{1}{2}$$

$f(n) = \text{sum of residues of } F(z)z^{n-1}$

$$\boxed{f(n) = \frac{(-1)^n}{4}[2n-1] + \frac{1}{2}}$$

5. **Using complex residue theorem evaluate** $Z^{-1}\left[\frac{9z^3}{(3z-1)^2(z-2)}\right]$**.**

Solution:

$$Z^{-1}\left[\frac{9z^3}{(3z-1)^2(z-2)}\right] = Z^{-1}\left[\frac{9z^3}{9(z-\frac{1}{3})^2(z-2)}\right] = Z^{-1}\left[\frac{z^3}{(z-\frac{1}{3})^2(z-2)}\right]$$

$$F(z) = \frac{z^3}{(z-\frac{1}{3})(z-2)}$$

$$F(z)z^{n-1} = \frac{z^3 z^{n-1}}{(z-\frac{1}{3})^2(z-2)}$$

$$F(z)z^{n-1} = \frac{z^{n+2}}{(z-\frac{1}{3})^2(z-2)}$$

Here $z=\frac{1}{3}$ are pole of order 2 and $z=2$ is simple pole.

1) $\operatorname{Res}\left[F(z)z^{n-1}\right]_{z=a} = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m F(z)z^{n-1}$ here $m=2$

$$\operatorname{Res}\left[F(z)\,z^{n-1}\right]_{z=\frac{1}{3}} = \lim_{z\to\frac{1}{3}} \frac{d}{dz}\left[\cancel{(z-\tfrac{1}{3})^2}\;\frac{z^{n+2}}{\cancel{(z-\tfrac{1}{3})^2}\,(z-2)}\right]$$

$$= \lim_{z\to\frac{1}{3}} \frac{d}{dz}\left[\frac{z^{n+2}}{z-2}\right]$$

$$= \lim_{z\to\frac{1}{3}}\left[\frac{(z-2)(n+2)z^{n+1} - z^{n+2}(1)}{(z-2)^2}\right]$$

$$= \lim_{z\to\frac{1}{3}}\left\{\frac{z^{n+1}\left[(z-2)(n+2)-z\right]}{(z-2)^2}\right\}$$

$$= \left[\frac{\left(\frac{1}{3}\right)^{n+1}\left[\left(\frac{1}{3}-2\right)(n+2)-\frac{1}{3}\right]}{\left(\frac{1}{3}-2\right)^2}\right]$$

$$\operatorname{Res}\left[F(z)\,z^{n-1}\right]_{z=\frac{1}{3}} = \frac{\left(\frac{1}{3}\right)^{n+1}\left[\left(\frac{-5(n+2)}{3}\right)-\frac{1}{3}\right]}{\left(\frac{-5}{3}\right)^2} = \frac{\left(\frac{1}{3}\right)^{n+1}\left(\frac{-5n-10-1}{3}\right)}{\frac{25}{9}}$$

$$= \frac{9}{25}\left(\frac{1}{3}\right)^n\left(\frac{1}{3}\right)\left(\frac{-5n-11}{3}\right) = \frac{-1}{25}\left(\frac{1}{3}\right)^n (5n+11)$$

$$\boxed{\operatorname{Res}\left[F(z)\,z^{n-1}\right]_{z=\frac{1}{3}} = \frac{-1}{25}\left(\frac{1}{3}\right)^n (5n+11)}$$

2) $\operatorname{Res}\left[F(z)\,z^{n-1}\right]_{z=2} = \lim_{z\to 2}\left(\cancel{(z-2)}\;\frac{z^{n+2}}{\cancel{(z-2)}\,(z-\frac{1}{3})^2}\right)$

$$\operatorname{Res}\left[F(z)\,z^{n-1}\right]_{z=2} = \frac{2^{n+2}}{\left(2-\frac{1}{3}\right)^2} = \frac{9}{25}2^{n+2}$$

$$\boxed{\operatorname{Res}\left[F(z)\,z^{n-1}\right]_{z=2} = \frac{9}{25}2^{n+2}}$$

$f(n) = \text{sum of residues of } F(z)z^{n-1}$

$$\boxed{f(n) = f(n) = \frac{9}{25}2^{n+2} + \frac{-1}{25}\left(\frac{1}{3}\right)^n (5n+11)}$$