Boundary Conditions For Electromagnetic Fields

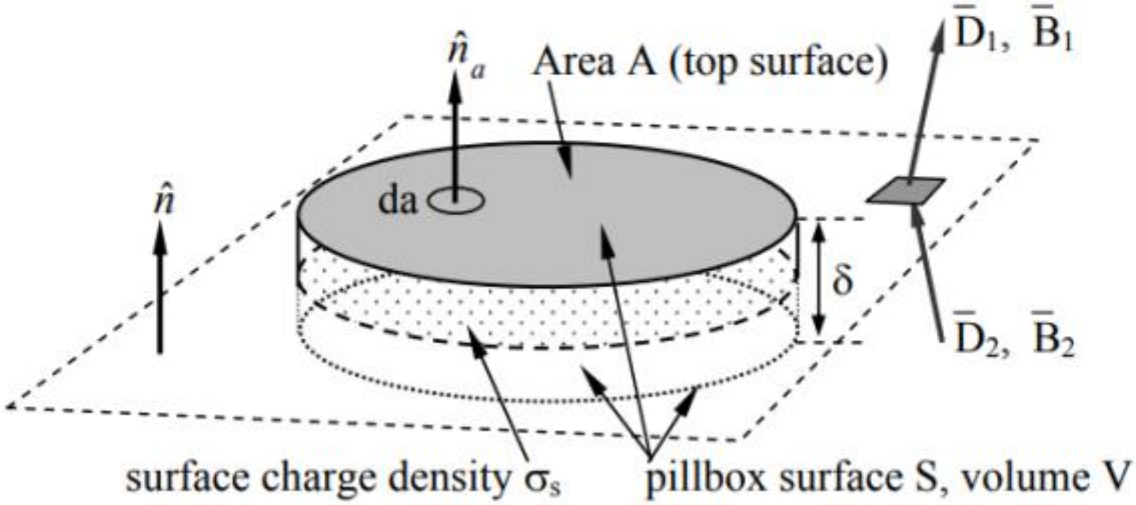
Introduction

Maxwell's equations characterize macroscopic matter by means of its permittivity ε, permeability μ, and conductivity σ, where these properties are usually represented by scalars and can vary among media. One result of these boundary conditions is that waves at boundaries are generally partially transmitted and partially reflected with directions and amplitudes that depend on the two media and the incident angles and polarizations. Static fields also generally have different amplitudes and directions on the two sides of a boundary. Some boundaries in both static and dynamic situations also possess surface charge or carry surface currents that further affect the adjacent fields. The boundary conditions governing the perpendicular components of and follow from the integral forms of Gauss's laws:

$$(\overline{\mathrm{D}} \bullet \hat{n})\mathrm{da} = \iiint_{\mathrm{V}} \rho \mathrm{dv} \qquad \text{(Gauss 's Law for } \overline{\mathrm{D}}\text{)}$$

$$(\overline{\mathrm{B}} \bullet \hat{n})\mathrm{da} = 0 \qquad \text{(Gauss 's Law for } \overline{\mathrm{B}}\text{)}$$

We may integrate these equations over the surface S and volume V of the thin infinitesimal pillbox illustrated in Figure. The pillbox is parallel to the surface and straddles it, half being on each side of the boundary. The thickness δ of the pillbox approaches zero faster than does its surface area S, where S is approximately twice the area A of the top surface of the box.



Elemental volume for deriving boundary conditions for perpendicular field components

Beginning with the boundary condition for the perpendicular component D , we integrate Gauss's law over the pillbox to obtain:

$$\left(\overline{\mathrm{D}} \bullet \hat{n}_a\right) \mathrm{da} \cong (\mathrm{D}_{1\perp} - \mathrm{D}_{2\perp})\, \mathrm{A} = \iiint_{\mathrm{V}} \rho \mathrm{dv} = \rho_s \mathrm{A}$$

where ρ is the surface charge density [Coulombs m]. The subscript s for surface charge ρ distinguishes it from the volume charge density ρ [C m]. The pillbox is so thin (δ → 0) that: 1) the contribution to the surface integral of the sides of the pillbox vanishes in comparison to the rest of the integral, and 2) only a surface charge q can be contained within it, where ρ = q/A = lim ρδ as the charge density ρ → ∞ and δ → 0. Thus the above equation becomes $D_1 - D_2 = \rho$, which can be written as:

$$\hat{n} \bullet \left(\overline{\mathrm{D}}_1 - \overline{\mathrm{D}}_2\right) = \rho_s \qquad \text{(boundary condition for } \overline{\mathrm{D}}_\perp\text{)}$$

where is the unit overlinetor normal to the boundary and points into medium 1. Thus the perpendicular component of the electric displacement overlinetor changes value at a boundary in accord with the surface charge density ρ . Because Gauss's laws are the same for electric and magnetic fields, except that there are no magnetic charges, the same analysis for the magnetic flux density in the second equation yields a similar boundary condition:

$$\hat{n} \bullet \left(\overline{\mathrm{B}}_1 - \overline{\mathrm{B}}_2\right) = 0 \qquad \left(\text{boundary condition for } \overline{\mathrm{B}}_\perp\right)$$

Thus the perpendicular component of must be continuous across any boundary.

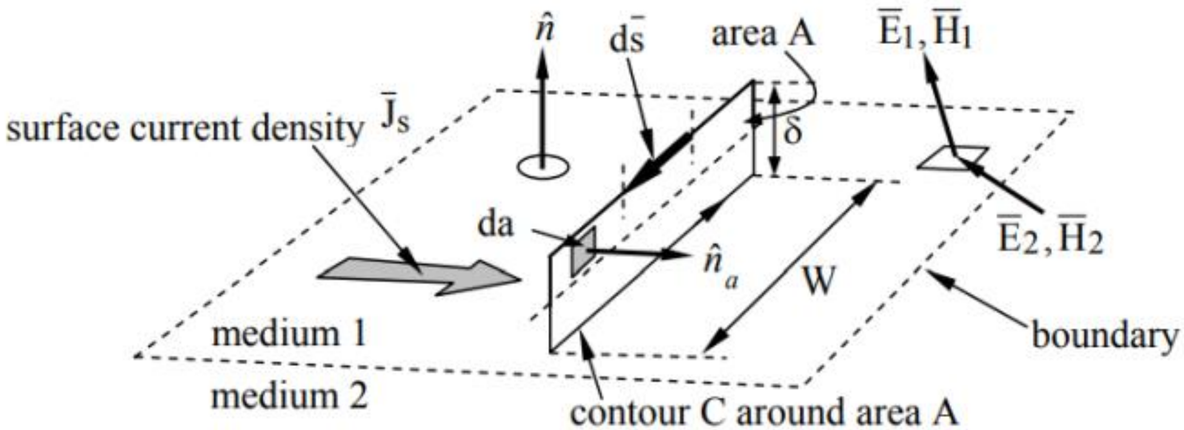
Boundary conditions for parallel field components

The boundary conditions governing the parallel components of and follow from Faraday's and Ampere's laws:

$$\oint_C \overline{E} \bullet d\overline{s} = -\frac{\partial}{\partial t} \int\int_A \overline{B} \bullet \hat{n} da \qquad (\text{Faraday's Law})$$

$$\oint_{\mathrm{C}} \overline{\mathrm{H}} \bullet \mathrm{d}\overline{\mathrm{s}} = \int\int_{\mathrm{A}} \left[\overline{\mathrm{J}} + \frac{\partial \overline{\mathrm{D}}}{\partial \mathrm{t}}\right] \bullet \hat{n} \mathrm{da} \qquad (\text{Ampere's } Law)$$

We can integrate these equations around the elongated rectangular contour C that straddles the boundary and has infinitesimal area A, as illustrated in the below figure. We assume the total height δ of the rectangle is much less than its length W, and circle C in a right hand sense relative to the surface normal.



Elemental contour for deriving boundary conditions for parallel field components

Beginning with Faraday's law, we find:

$$\oint_{\mathrm{C}} \overline{\mathrm{E}} \bullet \mathrm{d}\overline{\mathrm{s}} \cong \left(\overline{\mathrm{E}}_{1//} - \overline{\mathrm{E}}_{2//}\right) \mathrm{W} = -\frac{\partial}{\partial \mathrm{t}} \int\int_{\mathrm{A}} \overline{\mathrm{B}} \bullet \hat{n}_a \mathrm{da} \to 0$$

where the integral of over area A approaches zero in the limit where δ approaches zero too; there can be no impulses in . Since W ≠ 0, it follows from previous equation that E - E = 0, or more generally:

$$\hat{n} \times \left(\overline{\mathrm{E}}_1 - \overline{\mathrm{E}}_2\right) = 0 \qquad \left(\text{boundary condition for } \overline{\mathrm{E}}_{//}\right)$$

Thus the parallel component of must be continuous across any boundary. A similar integration of Ampere's law, under the assumption that the contour C is chosen to lie in a plane perpendicular to the surface current and is traversed in the right-hand sense, yields:

$$\oint_{\mathrm{C}} \overline{\mathrm{H}} \bullet \mathrm{d}\overline{\mathrm{s}} = \left(\overline{\mathrm{H}}_{1//} - \overline{\mathrm{H}}_{2//}\right) \mathrm{W}$$

$$= \int\int_{\mathrm{A}} \left[\overline{\mathrm{J}} + \frac{\partial \overline{\mathrm{D}}}{\partial \mathrm{t}}\right] \bullet \hat{n} \mathrm{da} \Rightarrow \int\int_{\mathrm{A}} \overline{\mathrm{J}} \bullet \hat{n}_a \mathrm{da} = \overline{\mathrm{J}}_{\mathrm{S}} \mathrm{W}$$

where we note that the area integral of approaches zero as $\delta \rightarrow 0$, but not the integral over the surface current , which occupies a surface layer thin compared to δ. Thus , or more generally:

$$\hat{n} \times \left(\overline{\mathrm{H}}_1 - \overline{\mathrm{H}}_2\right) = \overline{\mathrm{J}}_s \qquad \left(\text{boundary condition for } \overline{\mathrm{H}}_{//}\right)$$

where is defined as pointing from medium 2 into medium 1. If the medium is non conducting,
A simple static example illustrates how these boundary conditions generally result in fields on two sides of a boundary pointing in different directions. Consider the magnetic fields and illustrated in the below figure where , and both media are insulators so the surface current must be zero.

$$\mathrm{H}_{2\perp} = \mathrm{B}_{2\perp}/\mu_2 = \mathrm{B}_{1\perp}/\mu_2 = \mu_1 \mathrm{H}_{1\perp}/\mu_2$$

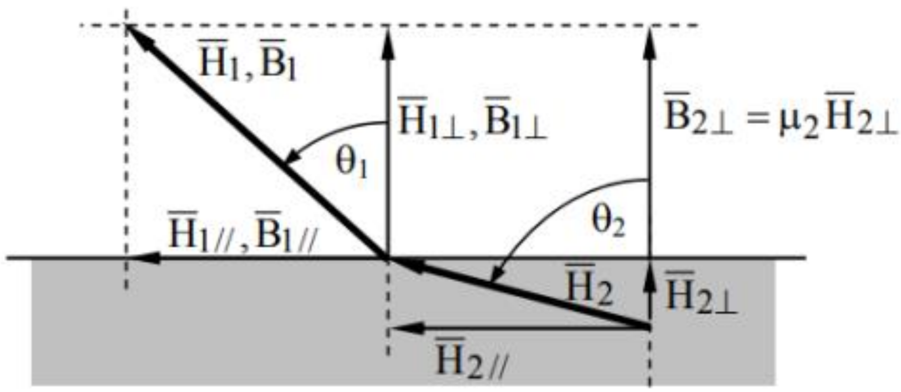


Figure 2.6.3: Static magnetic field directions at a boundary.

$$\theta_2 = \tan^{-1}\left(\left|\overline{\mathrm{H}}_{2//}\right| / \mathrm{H}_{2\perp}\right) = \tan^{-1}\left(\mu_2 \left|\overline{\mathrm{H}}_{1//}\right| / \mu_1 \mathrm{H}_{1\perp}\right) = \tan^{-1}\left[(\mu_2/\mu_1)\tan\theta_1\right]$$

Thus θ approaches 90 degrees when $\mu >> \mu$, almost regardless of θ , so the magnetic flux inside high permeability materials is nearly parallel to the walls and trapped inside, even when the field orientation outside the medium is nearly perpendicular to the interface. The flux escapes high-μ material best when $\theta \cong 90°$. This phenomenon is critical to the design of motors or other systems incorporating iron or nickel. If a static surface current flows at the boundary, then the relations between and are altered along with those for and . Similar considerations and methods apply to static electric fields at a boundary, where any static surface charge on the boundary alters the relationship between and . Surface currents normally arise only in non-static or “dynamic” cases.