UNIT-IV

FOURIER TRANSFORMS

Fourier sine and cosine transforms:

Fourier sine Transform

We know that the Fourier sine integral is

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin \lambda x \,.\, \int_0^{\infty} f(t) \sin\lambda t \; dt.d\lambda.$$

Replacing λ by s, we get

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \sin sx \,.\, \int_0^{\infty} f(t) \sin st \; dt. \; ds.$$

It follows that if

$$F_s(s) = \sqrt{(2/\pi)} \int_0^{\infty} f(t) \sin st \; dt..$$

i.e., $$F_s(s) = \sqrt{(2/\pi)} \int_0^{\infty} f(x) \sin sx \; dx. \text{ --------------------------------- (1)}$$

then $$f(x) = \sqrt{(2/\pi)} \int_0^{\infty} F_s(s) \sin sx \; ds. \text{------------------------------ (2)}$$

The function $F_s(s)$, as defined by (1), is known as the **Fourier sine transform** of f(x). Also the function f(x), as given by (2),is called the **Inverse Fourier sine transform** of $F_s(s)$.

Fourier cosine transform

Similarly, it follows from the Fourier cosine integral

$$f(x) = \frac{2}{\pi} \int_0^\infty \cos \lambda x \, . \int_0^\infty f(t) \cos\lambda t \; dt.d\lambda.$$

that if $\quad F_c(s) = \sqrt{(2/\pi)} \int_0^\infty f(x) \cos sx \; dx.$ -------------------------------- (3)

then $\quad f(x) = \sqrt{(2/\pi)} \int_0^\infty F_c(s) \cos sx \; ds.$ ------------------------------------- (4)

The function $F_c(s)$, as defined by (3), is known as the **Fourier cosine transform** of f(x). Also the function f(x), as given by (4),is called the **Inverse Fourier cosine transform** of $F_c(s)$.

Properties of Fourier sine and cosine Transforms

If $F_s(s)$ and $F_c(s)$ are the Fourier sine and cosine transforms of f(x) respectively, the following properties and identities are true.

(1) Linearity property

F_s [a f(x) + b g(x)] = a F_s { f(x) } + b F_s { g(x) }.

and F_c [a f(x) + b g(x)] = a F_c { f(x) } + b F_c { g(x) }.

(2) Change of scale property

F_s [f(ax)] = (1/a) F_s [s/a].

and F_c [f(ax)] = (1/a) F_c [s/a].

(3) Modulation Theorem

i. F_s [f(x) sinax] = (1/2) [F_c (s-a) - F_c (s+a)].

i. F_s [f(x) cosax] = (1/2) [F_s (s+a) + F_s (s-a)].

ii. $F_c[\, f(x) \cos ax \,] = (1/2)\, [\, F_c(s+a) + F_c(s-a) \,]$.

iii. $F_c[\, f(x) \sin ax \,] = (1/2)\, [\, F_s(s+a) - F_s(s-a) \,]$.

Proof

The Fourier sine transform of f(x)sinax is given by

$$F_s[\, f(x) \sin ax \,] = \sqrt{(2/\pi)} \int_0^\infty (f(x) \sin ax) \sin sx \, dx.$$

$$= (1/2) \sqrt{(2/\pi)} \int_0^\infty f(x) \, [\cos(s-a)x - \cos(s+a)x] \, dx.$$

$$= (1/2)\, [\, F_c(s-a) - F_c(s+a) \,].$$

Similarly, we can prove the results (ii), (iii) & (iv).

(4) Parseval's identity

$$\int_0^\infty F_c(s)\, G_c(s) \, ds = \int_0^\infty f(x)\, g(x) \, dx \, .$$

$$\int_0^\infty F_s(s)\, G_s(s) \, ds = \int_0^\infty f(x)\, g(x) \, dx \, .$$

$$\int_0^\infty \left| F_c(s) \right|^2 ds = \int_0^\infty \left| f(x) \right|^2 dx \, .$$

$$\int_0^\infty \left| F_s(s) \right|^2 ds = \int_0^\infty \left| f(x) \right|^2 dx \, .$$

Proof

$$\int_0^\infty F_c(s)\, G_c(s) \, ds = \int_0^\infty F_c(s) \left[\sqrt{(2/\pi)} \int_0^\infty g(t) \cos st \, dt \right] ds$$

$$= \int_0^\infty g(t) \left[\sqrt{(2/\pi)} \int_0^\infty F_c(s) \cos st \, ds \right] dt$$

$$= \int^\infty g(t)\, f(t) \, dt$$

i.e., $\int_0^\infty F_c(s)\, G_c(s)\, ds = \int_0^\infty f(x)\, g(x)\, dx$.

Similarly, we can prove the second identity and the other identities follow by settingg(x) = f(x) in the first identity.

Property (5)

If $F_s(s)$ and $F_c(s)$ are the Fourier sine and cosine transforms of f(x) respectively, then

(i) $F_s\{ x f(x) \} = - \dfrac{d}{ds} F_c(s)$.

(ii) $F_c\{ x f(x) \} = - \dfrac{d}{ds} F_s(s)$.

Proof

The Fourier cosine transform of f(x),

i.e., $F_c(s) = \sqrt{(2/\pi)} \int_0^\infty f(x) \cos sx\, dx.$

Differentiating w.r.t s, we get

$$\frac{d}{ds} [F_c(s)] = \sqrt{(2/\pi)} \int_0^\infty f(x) \{- x \sin sx \}\, dx.ds$$

$$= - \sqrt{(2/\pi)} \int_0^\infty (x f(x)) \sin sx\, dx.$$

$$= - F_s\{x f(x)\}$$

i.e., $F_s\{x f(x)\} = - \dfrac{d}{ds} \{ F_c(s) \}$

Similarly, we can prove

$F_c\{x f(x)\} = - \dfrac{d}{} \{ F_s(s) \}ds$

Example 8

Find the Fourier sine and cosine transforms of e^{-ax} and hence deduce the inversionformula.

The Fourier sine transform of f(x) is given by

$F_s\{ f(x) \} = \sqrt{(2/\pi)} \int_0^\infty f(x) \sin sx\, dx.$

Now , $F_s\{ e^{-ax} \} = \sqrt{(2/\pi)} \int_0^\infty e^{-ax} \sin sx \, dx.$

$$= \sqrt{(2/\pi)} \left\{ \frac{e^{-ax} (- a \sin sx - s \cos sx)}{a^2 + s^2} \right\}_0^\infty$$

$$= \sqrt{(2/\pi)} \frac{s}{a^2 + s^2} \quad \text{, if } a>0$$

The Fourier cosine transform of f(x) is given by

$$F_c\{ f(x) \} = \sqrt{(2/\pi)} \int_0^\infty f(x) \cos sx \, dx.$$

Now , $F_c\{ e^{-ax} \} = \sqrt{(2/\pi)} \int_0^\infty e^{-ax} \cos sx \, dx.$

$$= \sqrt{(2/\pi)} \left\{ \frac{e^{-ax} (- a \cos sx + s \sin sx)}{a^2 + s^2} \right\}_0^\infty$$

$$= \sqrt{(2/\pi)} \frac{a}{a^2 + s^2} \quad \text{, if } a>0$$

Example 9

Find the Fourier cosine transform of $f(x) = \begin{cases} x, & \text{for } 0<x<1 \\ 2 - x, & \text{for } 1<x<2 \\ 0, & \text{for } x>2 \end{cases}$

The Fourier cosine transform of f(x),

$$\text{i.e., } F_c\{ f(x) \} = \sqrt{(2/\pi)} \int_0^1 x \cos sx \, dx. + \sqrt{(2/\pi)} \int_1^2 (2 - x) \cos sx \, dx.$$

$$= \sqrt{(2/\pi)} \int_0^1 x \; d\left(\frac{\sin sx}{s}\right) + \sqrt{(2/\pi)} \int_1^2 (2 - x) \; d\left(\frac{\sin sx}{s}\right)$$

$$= \sqrt{(2/\pi)} \left\{ x\left(\frac{\sin sx}{s}\right) - (1) \left(- \frac{\cos sx}{s^2}\right)\right\}_0^1 + \sqrt{(2/\pi)} \left\{ (2 - x)\left(\frac{\sin sx}{s}\right) - (- 1) \left(- \frac{\cos sx}{s^2}\right)\right\}_1^2$$

$$= \sqrt{(2/\pi)}\left\{\left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2}\right) + \left(-\frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2}\right)\right\}$$

$$= \sqrt{(2/\pi)}\left(\frac{2\cos s}{s^2} - \frac{\cos 2s}{s^2} - \frac{1}{s^2}\right)$$

Example 10

Find the Fourier sine transform of e^{-x}. Hence show that $\int_0^{\infty} \frac{x \sin mx}{1+x^2}\, dx = \frac{\pi e^{-m}}{2}$, m>0.

The Fourier sine transform of f(x) is given by

$$F_s\{ f(x) \} = \sqrt{(2/\pi)} \int_0^{\infty} f(x) \sin sx \; dx.$$

$$= \sqrt{(2/\pi)} \int_0^{\infty} e^{-x} \sin sx \; dx.$$

$$= \sqrt{(2/\pi)} \left\{ \frac{e^{-x}(-\sin sx - s\cos sx)}{1+s^2} \right\}_0^{\infty}$$

$$= \sqrt{(2/\pi)} \frac{s}{1+s^2}.$$

Using inversion formula for Fourier sine transforms, we get

$$\sqrt{(2/\pi)} \int_0^{\infty} \sqrt{(2/\pi)} \frac{s}{1+s^2} \sin sx \; ds. = e^{-x}$$

Replacing x by m,

$$e^{-m} = (2/\pi) \int_0^{\infty} \frac{s \sin ms}{1+s^2}\, ds$$

$$\int^{\infty} \text{x sinmx}$$

$$= (2/\pi) \int_0 \frac{\qquad}{1+x^2}\, dx$$

Hence, $$\int_0^\infty \frac{x \sin mx}{1+x^2}\, dx = \frac{\pi e^{-m}}{2}$$

Example 11

Find the Fourier sine transform of $\dfrac{x}{a^2+x^2}$ and the Fourier cosine transform of $\dfrac{1}{a^2+x^2}$.

To find the Fourier sine transform of $\dfrac{x}{a^2+x^2}$,

We have to find $F_s\{ e^{-ax} \}$.

Consider, $$F_s\{ e^{-ax} \} = \sqrt{(2/\pi)} \int_0^\infty e^{-ax} \sin sx \, dx.$$

$$= \sqrt{(2/\pi)} \frac{s}{a^2+s^2} .$$

Using inversion formula for Fourier sine transforms, we get

$$e^{-ax} = \sqrt{(2/\pi)} \int_0^\infty \sqrt{(2/\pi)} \left\{ \frac{s}{a^2+s^2} \right\} \sin sx \, ds.$$

i.e., $$\int_0^\infty \frac{s \sin sx}{s^2+a^2}\, ds = \frac{\pi e^{-ax}}{2} \quad , a>0$$

Changing x by s, we get

$$\int_0^\infty \frac{x \sin sx}{x^2+a^2}\, dx = \frac{\pi e^{-as}}{2} \text{-----------------------------}(1)$$

Now $$F_s\left(\frac{x}{x^2+a^2} \right) = \sqrt{(2/\pi)} \int_0^\infty \frac{x}{x^2+a^2} \sin sx \, dx$$

. $$= \sqrt{(2/\pi)} \frac{\pi e^{-as}}{2} , \qquad \text{using (1)}$$

$$= \sqrt{(\pi/2)}\, e^{-as}$$

Similarly,for finding the Fourier cosine transform of $\dfrac{1}{a^2 + x^2}$, we have to find $F_c\{e^{-ax}\}$.

Consider , $F_c\{ e^{-ax} \} = \sqrt{(2/\pi)} \int_0^\infty e^{-ax} \cos sx \; dx.$

$$= \sqrt{(2/\pi)} \frac{a}{a^2 + s^2}.$$

Using inversion formula for Fourier cosine transforms, we get

$$e^{-ax} = \sqrt{(2/\pi)} \int_0^\infty \left\{ \sqrt{(2/\pi)} \frac{a}{a^2 + s^2} \right\} \cos sx \, ds.$$

i.e., $$\int_0^\infty \frac{\cos sx}{s^2 + a^2} ds = \frac{\pi e^{-ax}}{2a}$$

Changing x by s, we get

$$\int_0^\infty \frac{\cos sx}{x^2 + a^2} dx = \frac{\pi e^{-as}}{2a} \text{-----------------------------------(2)}$$

Now, $$F_c\left(\frac{1}{x^2 + a^2}\right) = \sqrt{(2/\pi)} \int_0^\infty \frac{1}{x^2 + a^2} \cos sx \, dx$$

$$= \sqrt{(2/\pi)} \frac{\pi e^{-as}}{2a}, \quad \text{using (2)}$$

$$= \sqrt{(\pi/2)} \frac{e^{-as}}{a}$$

Example 12

Find the Fourier cosine transform of $e^{-a^2 x^2}$ and hence evaluate the Fourier sine transform of $xe^{-a^2 x^2}$.

The Fourier cosine transform of $e^{-a^2x^2}$ is given by

$$F_c\{e^{-a^2x^2}\} = \sqrt{(2/\pi)} \int_0^{\infty} e^{-a^2x^2} \cos sx \; dx$$

$$= \text{Real part of } \sqrt{(2/\pi)} \int_0^{\infty} e^{-a^2x^2} e^{isx} \; dx$$

$$= \text{Real part of } \frac{1}{.\sqrt{2}.} e^{-s^2/4a^2}. \quad \text{(Refer example (4) of section 4.4)a}$$

$$= \frac{1}{a.\sqrt{2}.} e^{-s^2/4a^2}. \text{------------------------ (i)}$$

But , $F_s\{x f(x)\} = - \dfrac{d}{ds} F_c(s)$

$$\therefore F_s\{x e^{-a^2x^2}\} = - \frac{d}{ds}\left\{ \frac{1}{a\sqrt{2}} e^{-s^2/4a^2} \right. \quad \text{, by (1)}$$

$$= - \frac{1}{\sqrt{2}} e^{-s^2/4a^2} (-s/2a^2).a$$

$$= \frac{s}{2\sqrt{2}.a^3} e^{-s^2/4a^2}.$$

$F_c[1/\sqrt{x}] = 1/\sqrt{s}$

and $F_s[1/\sqrt{x}] = 1/\sqrt{s}$

This shows that $1/\sqrt{x}$ is self-reciprocal.