# UNIT-IV

# FOURIER TRANSFORMS

## Introduction

This unit starts with integral transforms and presents three well-known integral transforms, namely, Complex Fourier transform, Fourier sine transform, Fourier cosine transform and their inverse transforms. The concept of Fourier transforms will be introduced after deriving the Fourier Integral Theorem. The various properties of these transforms and many solved examples are provided in this chapter. Moreover, the applications of Fourier Transforms in partial differential equations are many and are not included here because it is a wide area and beyond the scope of the book.

### Integral Transforms

The **integral transform** $\tilde{f}(s)$ of a function f(x) is defined by

$$\tilde{f}(s) = \int^{b} f(x)\ K(s,x)\, dx, a$$

if the integral exists and is denoted by I{f(x)}. Here, K(s,x) is called the **kernel** of the transform. The kernel is a known function of „s‟ and „x‟. The function f(x) is called the

**inverse transform**

of $\tilde{f}(s)$. By properly selecting the kernel in the definition of general integral transform,we get various integral transforms.

The following are some of the well-known transforms:

**(i) Laplace Transform**

$$L\{f(x)\} = \int_0^{\infty} f(x)\ e^{-sx}\ dx$$

**(ii) Fourier Transform**

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

**(iii) Mellin Transform**

$$M\{f(x)\} = \int_0^{\infty} f(x)\, x^{s-1}\, dx$$

## (iv) Hankel Transform

$$H_n\{f(x)\} = \int_0^{\infty} f(x)\, x\, J_n(sx)\, dx,$$

where $J_n(sx)$ is the Bessel function of the first kind and order „n‟.

FOURIER INTEGRAL THEOREM

If f(x) is defined in the interval (-ℓ,ℓ), and the following conditions

(i) f(x) satisfies the Dirichlet‟s conditions in every interval (-ℓ,ℓ),

(ii) $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges, i.e. f(x) is absolutely integrable in (-∞,∞)

are true, then $f(x) = (1/\pi)\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda$.

Consider a function f(x) which satisfies the Dirichlet‟s conditions in every interval (-ℓ,ℓ)so that, we have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell} \quad \text{------(1)}$$

where $a_0 = \dfrac{1}{\ell}\int_{-\ell}^{\ell} f(t)\,dt$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(t)\cos(n\pi t/\ell)\,dt$$

and $b_n = \dfrac{1}{\ell}\int_{-\ell}^{\ell} f(t)\sin(n\pi t/\ell)\,dt$

Substituting the values of $a_0$, $a_n$ and $b_n$ in (1), we get

$$f(x) = \frac{1}{2\ell}\int_{-\ell}^{\ell} f(t)\,dt + \frac{1}{\ell}\sum_{n=1}^{\infty}\int_{-\ell}^{\ell} f(t)\cos\frac{n\pi(t-x)}{\ell}\,dt \quad \text{-------(2)}$$

Since,
$$\left| \frac{1}{2\ell} \int_{-\ell}^{\ell} f(t)\, dt \right| \le \frac{1}{2\ell} \int_{-\ell}^{\ell} |f(t)|\, dt ,$$

then by assumption (ii), the first term on the right side of (2) approaches zero as $\ell \to \infty$.

As $\ell \to \infty$, the second term on the right side of (2) becomes

$$\lim_{\ell \to \infty} \frac{1}{\ell} \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \frac{n\pi(t-x)}{\ell}\, dt$$

$$= \lim_{\Delta\lambda \to 0} \frac{1}{\pi} \sum_{n=1}^{\infty} \Delta\lambda \int_{-\infty}^{\infty} f(t) \cos\{ n\,\Delta\lambda\,(t-x)\}\, dt \text{ ,on taking } (\pi/\ell) = \Delta\lambda.$$

By the definition of integral as the limit of sum and $(n\pi / \ell) = \lambda$ as $\ell \to \infty$, the secondterm of (2) takes the form

$$\frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda (t-x)\, dt\, d\lambda ,$$

Hence as $\ell \to \infty$, (2) becomes

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos \lambda (t-x)\, dt\, d\lambda \text{ ------------------ } (3)$$

which is known as the **Fourier integral** of f(x).

**Note:**

When f(x) satisfies the conditions stated above, equation (3) holds good at a point of continuity. But at a point of discontinuity, the value of the integral is (1/2) [f(x+0) +f(x-0)] as in the case of Fourier series.

**Fourier sine and cosine Integrals**

The Fourier integral of f(x) is given by

$$f(x) = \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \cos\lambda (t-x)\, dt\, d\lambda$$

$$= \frac{1}{\pi} \int_{0}^{\infty} \int_{-\infty}^{\infty} f(t) \{ \cos\lambda t \,.\, \cos\lambda x + \sin\lambda t \,.\, \sin\lambda x \}\, dt\, d\lambda$$

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$$= \frac{}{\pi}\int_{0} \cos\lambda x \int_{-\infty} f(t)\cos\lambda t \, dt \, d\lambda \; + \; \frac{}{\pi}\int_{0} \sin\lambda x \int_{-\infty} f(t)\sin\lambda t \, dt \, d\lambda \text{ -------- (4)}$$

When f(x) is an odd function, f(t) cosλt is odd while f(t) sinλt is even. Then the firstintegral of (4) vanishes and, we get

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \sin\lambda x \int_{-\infty}^{\infty} f(t)\sin\lambda t \, dt \, d\lambda \text{ -------------- (5)}$$

which is known as the **Fourier sine integral.**

Similarly, when f(x) is an even function, (4) takes the form

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty} \cos\lambda x \int_{-\infty}^{\infty} f(t)\cos\lambda t \, dt \, d\lambda \text{ -------------- (6)}$$

which is known as the **Fourier cosine integral.**

**Complex form of Fourier Integrals**

The Fourier integral of f(x) is given by

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t) \cos \lambda(t - x) \, dt \, d\lambda$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t) \int_{0}^{\infty} \cos \lambda(t - x) \, d\lambda \, dt$$

Since cos λ(t – x) is an even function of λ, we have by the property of definite integrals

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t) \; (1/2) \int_{-\infty}^{\infty} \cos \lambda(t - x) \, d\lambda \, dt$$

$$\text{i.e.,} \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) \cos \lambda(t - x) \, dt \, d\lambda \text{ ----------------- (7)}$$

Similarly, since sin λ(t – x) is an odd function of λ, we have

$$0 = \frac{}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) \sin \lambda(t-x)\, dt\, d\lambda \text{------------------}(8)$$

Multiplying (8) by „i ‟ and adding to (7), we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{i\lambda(t-x)}\, dt\, d\lambda \text{ --------------------- }(9)$$

which is the **complex form of the Fourier integral.**