BUCKINGHAM PI THEOREM - DIMENSIONLESS PARAMETERS - SIMILITUDE AND MODEL STUDIES DISTORTED AND UNDISTORTED MODEL

If the number of variables involved in a physical phenomenon are known, then the relation among the variables can be determined by the following two methods.

- Reyleigh's method
- Buckingham's Pi-theorem

Reyleigh's method

This method is used for determining the expression for a variable which depends upon maximum three or four variables only. If the number of independent variables is more than five then it becomes difficult to find expression for dependent variable.

Let $X_1, X_2, X_3 --------X_n$ are the variables involved in a physical problem. Let X_1 be the dependent variable and $X_2, X_3 --------X_n$ are independent variable upon which X_1 depends.

$$X_1 = f(X_2, X_3 --------X_n)$$

i.e $f_1(X_1, X_2, X_3 --------X_n) = 0$ -------------------------- (i)

Buckingham's Pi-theorem

If there are n variables (independent and dependent) in a physical phenomenon and these variables contain m fundamental dimensions (M,L,T) then the variables are arranged into (n-m) dimensionless terms. Each term is called π term.

Let X_1, X_2,X_3 – – – – – – – – –X_n , , are the variables involved in a physical problem. Let X1 be the dependent variable and X_2, X_3 – – – – – – – – –X_n , are independent variable upon which X_1 depends.

$$X_1 = f(X_2, X_3 --------X_n)$$

i.e $f_1(X_1, X_2, X_3 --------X_n) = 0$ -------------------------- (i)

Equation (i) is dimensionally homogeneous equation. It contains n variables. If there are m fundamental dimensions then according to Buckingham' π – Theorem, eqn.(i) can be written in terms of number of dimensionless groups or π – terms in which number of π – terms is equal to (n-m). Hence eqn.(i) becomes

$f(\pi_1, \pi_2, \pi_3 --------\pi_{n-m}) = 0$ ------------------------------------ (ii)

Each π term is dimensionless and independent of the system. Division or multiplication by a constant does not change the character of the π – term. Each π – term contains m+1 variables, where m is number of fundamental dimensions and is also called repeating variables. Let in the above case X_2, X_3 ,X_4 are repeating variables if fundamental dimension m (M, L, T) = 3 then each π – term is written as

$$\pi_1 = X_2^{a_1} X_3^{b_1} X_4^{c_1} X_1$$
$$\pi_2 = X_2^{a_2} X_3^{b_2} X_4^{c_2} X_1$$

$$\pi_{n-m} = X_2^{a_{n-m}} X_3^{b_{n-m}} X_4^{c_{n-m}} X_1 \text{-------------------------- (iii)}$$

Each term is solved by the principle of dimensional homogeneity and values of a_1 ,b_1 ,c_1 etc are obtained. These values are substituted in the eqn. (iii) and values of $\pi_1\ \pi_2\ \pi_3$– – – – – – –π_{n-m} , , are obtained. These values are substituted in eqn. (ii). The final equation for the phenomenon is obtained by expressing any one of the π – terms as a function of others as

$$\pi_1 = \phi(\pi_2, \pi_3 \text{--------} \pi_{n-m}) = 0$$
$$\pi_2 = \phi(\pi_1, \pi_3 \text{--------} \pi_{n-m}) = 0$$

Method of selecting repeating variable:

1. As far as possible dependent variable should not be selected as repeating variable.

2. Repeating variables should be selected in such a way that one variable contains geometric property (such as length l , diameter d, height H etc), other variable contains flow properties (such as velocity, acceleration etc.) and the third variable contains fluid properties (such as viscosity, density etc)

3. Selected repeating variable should not form dimensionless group.

4. Repeating variables together must have same number of fundamental dimensions.

5. No two repeating variables should have the same dimension. For most of the fluidmechanics problems the choice for the repeating variable may be
(i) d,γ ,ρ (ii) l,γ ,ρ (iii) l,γ ,μ (iv) d,γ ,μ

PROBLEM 1: A partially submerged body is towed in water. The resistance R to its motion depends on the density ρ , viscosity μ of water, length L of the body, velocity V of the body and acceleration g due to gravity. Show that the resistance to the motion can be expressed in the form of

$$R = \rho L^2 V^2\ \phi\left[\left(\frac{\mu}{\rho V L}\right)\cdot\left(\frac{lg}{V^2}\right)\right].$$

Soln. The resistance R depends on ρ,μ,L,V, g

$$R = K\rho^a \cdot \mu^b \cdot l^c \cdot V^d \cdot g^e \qquad \text{...}(i)$$

Substituting the dimensions on both sides of the equation (*i*),

$$MLT^{-2} = K(ML^{-3})^a \cdot (ML^{-1}T^{-1})^b \cdot L^c \cdot (LT^{-1})^d \cdot (LT^{-2})^e$$

Equating the powers of M, L, T on both sides

Power of M, $1 = a + b$

Power of L, $1 = -3a - b + c + d + e$

Power of T, $-2 = -b - d - 2e$.

There are 5 unknowns and 3 equations. Expressing the three unknowns in terms of two unknowns (μ and g) . Hence express a, c and d in terms of b and e. Solving we get

$$a = 1 - b$$
$$d = 2 - b - 2e$$
$$c = 1 + 3a + b - d - e = 1 + 3(1 - b) + b - (2 - b - 2e) - e$$
$$= 1 + 3 - 3b + b - 2 + b + 2e - e = 2 - b + e.$$

Substituting these values in equation (*i*), we get

$$R = K\rho^{1-b} \cdot \mu^b \cdot l^{2-b+e} \cdot V^{2-b-2e} \cdot g^e$$
$$= K\rho l^2 \cdot V^2 \cdot (\rho^{-b}\mu^b l^{-b} V^{-b}) \cdot (l^e \cdot V^{-2e} \cdot g^e)$$

$$= K\rho l^2 V^2 \cdot \left(\frac{\mu}{\rho V l}\right)^b \cdot \left(\frac{lg}{V^2}\right)^e$$

$$= \mathbf{K\rho l^2 V^2 \phi\left[\left(\frac{\mu}{\rho V l}\right) \cdot \left(\frac{lg}{V^2}\right)\right]}. \quad \textbf{Ans.}$$

PROBLEM 2: The resisting force R of a supersonic plane during flight can be considered as dependent upon the length of the aircraft L, velocity V, air viscosity μ, air density , and bulk modulus of air k. Express the functional relationship between the variables and the resisting force.

Solution. The resisting force R depends upon

(*i*) density, l,
(*ii*) velocity, V,
(*iii*) viscosity, μ,
(*iv*) density, ρ,
(*v*) Bulk modulus, K.

$$\therefore \qquad R = Al^a \cdot V^b \cdot \mu^c \cdot \rho^d \cdot K^e \qquad \text{...}(i)$$

Substituting the dimensions on both sides of the equation (*i*),

$$MLT^{-2} = AL^a \cdot (LT^{-1})^b \cdot (ML^{-1}T^{-1})^c \cdot (ML^{-3})^d \cdot (ML^{-1}T^{-2})^e$$

Equating the powers of M, L, T on both sides,

Power of M, $1 = c + d + e$

Power of L, $1 = a + b - c - 3d - e$

Power of T, $-2 = -b - c - 2e$.

There are five unknowns but equations are only three. Expressing the three unknowns in terms of two unknowns (μ and K).

$\therefore$ Express the values of a, b and d in terms of c and e.

Solving,

$$d = 1 - c - e$$
$$b = 2 - c - 2e$$
$$a = 1 - b + c + 3d + e = 1 - (2 - c - 2e) + c + 3(1 - c - e) + e$$
$$= 1 - 2 + c + 2e + c + 3 - 3c - 3e + e = 2 - c.$$

Substituting these values in (i), we get

$$R = A\, l^{2-c} \,.\, V^{2-c-2e} \,.\, \mu^c \,.\, \rho^{1-c-e} \,.\, K^e$$

$$= A\, l^2 \,.\, V^2 \,.\, \rho(l^{-c}\, V^{-c}\, \mu^c\, \rho^{-c}) \,.\, (V^{-2e} \,.\, \rho^{-e} \,.\, K^e)$$
$$= A\, l^2 V^2 \rho \left(\frac{\mu}{\rho VL}\right)^c \cdot \left(\frac{K}{\rho V^2}\right)^e$$

PROBLEM 3:Using Buckingham' π – Theorem show that velocity through circular orifice is given by

$$V = \sqrt{2gH}\, \phi\left(\frac{D}{H}, \frac{\mu}{\rho VH}\right),$$

where H is head causing flow, D is diameter of the orifice, μ is coefficient viscosity, ρ

Solution. Given :

V is a function of H, D, μ, ρ and g

$$\therefore \qquad V = f(H, D, \mu, \rho, g) \text{ or } f_1(V, H, D, \mu, \rho, g) = 0$$

s mass density and g is acceleration due to gravity

$\therefore$ Total number of variable, $n = 6$ ...(*i*)

Writing dimension of each variable, we have

$$V = LT^{-1},\ H = L,\ D = L,\ \mu = ML^{-1}T^{-1},\ \rho = ML^{-3},\ g = LT^{-2}.$$

Thus number of fundamental dimensions, $m = 3$

$\therefore$ Number of π-terms $= n - m = 6 - 3 = 3.$

Equation (*i*) can be written as $f_1(\pi_1, \pi_2, \pi_3) = 0$ ...(*ii*)

Each π-term contains $m + 1$ variables, where $m = 3$ and is also equal to repeating variables. Here V is a dependent variable and hence should not be selected as repeating variable. Choosing H, g, ρ as repeating variable, we get three π-terms as

$$\pi_1 = H^{a_1} \cdot g^{b_1} \cdot \rho^{c_1} \cdot V$$

$$\pi_2 = H^{a_2} \cdot g^{b_2} \cdot \rho^{c_2} \cdot D$$

$$\pi_3 = H^{a_3} \cdot g^{b_3} \cdot \rho^{c_3} \cdot \mu$$

First π-term $\quad \pi_1 = H^{a_1} \cdot g^{b_1} \cdot \rho^{c_1} \cdot V$

Substituting dimensions on both sides

$$M^0L^0T^0 = L^{a_1} \cdot (LT^{-2})^{b_1} \cdot (MT^{-3})^{c_1} \cdot (LT^{-1})$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 0 = c, \quad \therefore \quad c_1 = 0$

Power of L, $\quad 0 = a_1 + b_1 - 3c_1 + 1, \quad \therefore \quad a_1 = -b_1 + 3c_1 - 1 = \frac{1}{2} - 1 = -\frac{1}{2}$

Power of T, $\quad 0 = -2b_1 - 1, \quad \therefore \quad b_1 = -\frac{1}{2}$

Substituting the values of a_1, b_1 and c_1 in π_1,

$$\pi_1 = H^{-\frac{1}{2}} \cdot g^{-\frac{1}{2}} \cdot \rho^0 \cdot V = \frac{V}{\sqrt{gH}}.$$

Second π-term $\quad \pi_2 = H^{a_2} \cdot g^{b_2} \cdot \rho^{c_2} \cdot D$

Substituting the dimensions on both sides,

$$M^0L^0T^0 = L^{a_2} \cdot (LT^{-2})^{b_2} \cdot (ML^{-3})^{c_2} \cdot L$$

Equating the powers of M, L, T,

Power of M, $\quad 0 = c_2 \quad \therefore \quad c_2 = 0$

Power of L, $\quad 0 = a_2 + b_2 - 3c_2 + 1,\ a_2 = -b_2 + 3c_2 - 1 = -1$

Power of T, $\quad 0 = -2b_2, \quad \therefore \quad b_2 = 0$

Substituting the values of a_2, b_2, c_2 in π_2,

$$\pi_2 = H^{-1} \cdot g^0 \rho^0 \cdot D = \frac{D}{H}.$$

Third π-term $\quad \pi_3 = H^{a_3} \cdot g^{b_3} \cdot \rho^{c_3} \cdot \mu$

Substituting the dimensions on both sides

$$M^0L^0T^0 = L^{a_3} \cdot (LT^{-2})^{b_3} \cdot (ML^{-3})^{c_3} \cdot ML^{-1}T^{-1}$$

Equating the powers of M, L, T on both sides

Power of M, $\quad 0 = c_3 + 1, \quad \therefore \quad c_3 = -1$

Power of L, $\quad 0 = a_3 + b_3 - 3c_3 - 1, \quad \therefore \quad a_3 = -b_3 + 3c_3 + 1 = \frac{1}{2} - 3 + 1 = -\frac{3}{2}$

Power of T, $\quad 0 = -2b_3 - 1, \quad \therefore \quad b_3 = -\frac{1}{2}$

Substituting the values of a_3, b_3 and c_3 in π_3,

$$\pi_3 = H^{-3/2} \cdot g^{-1/2} \cdot \rho^{-1} \cdot \mu = \frac{\mu}{H^{3/2}\rho\sqrt{g}}$$

$$= \frac{\mu}{H\rho\sqrt{gH}} = \frac{\mu V}{H\rho V\sqrt{gH}} \qquad \text{[Multiply and Divide by } V\text{]}$$

$$= \frac{\mu}{H\rho V} \cdot \pi_1 \qquad \left\{\because \quad \frac{V}{\sqrt{gH}} = \pi_1\right\}$$

Substituting the values of π_1, π_2 and π_3 in equation (ii),

$$f_1\left(\frac{V}{\sqrt{gH}}, \frac{D}{H}, \pi_1 \frac{\mu}{H\rho V}\right) = 0 \text{ or } \frac{V}{\sqrt{gH}} = \phi\left[\frac{D}{H}, \pi_1 \frac{\mu}{H\rho V}\right]$$

or

$$V = \sqrt{2gH}\ \phi\left[\frac{D}{H}, \frac{\mu}{\rho VH}\right]. \textbf{ Ans.}$$

Multiplying by a constant does not change the character of π-terms.

MODEL STUDIES

Model: Model is the small scale replica of the actual structure or machine. It is not necessary that models should be smaller than the prototypes (although in most of the cases it is), they may be larger than the prototypes.

Prototype: The actual structure or machine

Model analysis: Model analysis is the study of models of actual machine.

Advantages:

• The performance of the machine can be easily predicted, in advance.

• With the help of dimensional analysis, a relationship between the variables influencing a flow problem in terms of dimensional parameters is obtained. This relationship helps in conducting tests on the model.

• The merits of alternative designs can be predicted with the help of model testing. The most economical and safe design may be, finally, adopted.

Type of forces acting in the moving fluid

Inertial force: it is equal to the mass and acceleration of the moving fluid.

$$F_i = \rho A V^2$$

Viscous force: it is equal to the shear stress due to viscosity and surface area of the flow.It present in the flow problems where viscosity is having an important role to play.

$$F_v = \tau A = \mu \frac{du}{dy} A = \mu \frac{U}{d} A$$

Gravity force: product of mass and acceleration due to gravity.

$$F_g = \rho A L g$$

Pressure force: product of pressure intensity and flow area.

$$F_p = pA$$

Surface tension force: product of surface tension and the length of the surface of the flowing fluid.

$$F_s = \sigma d$$

Elastic force: product of elastic stress and area of the flow.

$$F_e = Elastic\ stress \times Area = KA$$

Classification of model

• Undistorted models: are those models which are geometrically similar to their prototype. In other words the scale ratio for the linear dimensions of the model and itsprototype are the same.

• Distorted models: are those models which are geometrically not similar to its prototype. In other words the scale ratio for the linear dimensions of the model and itsprototype are not same.

For example river: If the horizontal and vertical scale ratios for the model and the prototype are same then it is undistorted model. In this case the depth of the water in the model becomes very small which may not be measured accurately.

Thus for cases distorted model is useful.
The followings are the advantages of distorted models

- ✓ The vertical dimension of the model can be accurately measured
- ✓ The cost of the model can be reduced
- ✓ Turbulent flow in the model can be maintained

Though there are some advantage of distorted models, however the results of suchmodels cannot be directly transferred to prototype