

LINEARLY INDEPENDENCE AND LINEARLY DEPENDENCE

Linearly dependent set

A subset S of a vector space is called linearly dependent if there is a finite number of distinct vectors v_1, v_2, \dots, v_n in S and scalars $\alpha_1, \alpha_2, \dots$, zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

Linearly independent set

A subset S of a vector space that is not linearly dependent is called independent. i.e., A subset S of a vector space is called linearly independent if there exists a finite number of distinct vectors v_1, v_2, \dots, v_n in S and scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ such

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0. \text{ Implies } \alpha_1 = \alpha_2 = \dots = \alpha_n = 0$$

Note:

- Any set of vectors which contains zero vectors is linearly dependent
- In R^2 any two straight lines which are not parallel are linearly independent
- In R^2 any two straight lines which are parallel are linearly dependent
- In R^2 any three vectors are linearly dependent therefore any set of n in the R^m are linearly dependent if $n > m$.

Theorem 1.16: $\{0\}$ is a dependent set

Proof: Let V be a vector space over F

Let $v_1 = 0$

Therefore $\alpha_1 v_1 = 0 \Rightarrow \alpha_1 \neq 0$

$\therefore \{0\}$ is linearly dependent.

Theorem 1.17: A singleton non zero vector is linearly independent set

Proof: Let V be a vector space over F

Let $v_1 \neq 0 \in V$

Therefore $\alpha_1 v_1 = 0 \Rightarrow \alpha_1 = 0$

$\therefore \{v_1\}$ is linearly independent.

Theorem 1.18: Any subset of a linearly independent set is linearly independent.

Proof:

Let V be a vector space over a field F .

Let $S = \{v_1, v_2, \dots, v_n\}$ be a linearly independent set.

Let $S_1 = \{v_1, v_2, \dots, v_m\}$ be a subset of S , where $m < n$.

Suppose S_1 is a linearly dependent set. Then there exist $\alpha_1, \alpha_2, \dots, \alpha_m$ in F not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m = 0$$

Hence $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_m v_m + 0v_{m+1} + \dots + 0v_n = 0$ with $\alpha_1, \alpha_2, \dots, \alpha_m$ in F not all zero.

Therefore $\{v_1, v_2, \dots, v_m, v_{m+1}, \dots, v_n\}$ is a linearly dependent set of V i.e., S is a linearly dependent set of V , which is a contradiction.

Therefore S_1 is linearly independent.

Theorem 1.19: Any set containing a linearly dependent set is also linearly dependent

OR

Any super set of a linearly dependent set is linearly dependent set

Proof: Let V a vector space over F .

Let S be a linearly dependent set of V ...

Then there exists scalar $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ not all zero such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

now consider the super set $S_1 = \{v_1, v_2, \dots, v_n, v_{n+1}\}$

Then we have $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n + 0v_{n+1} = 0$ with at least one $\alpha_i \neq 0$

$\therefore S_1$ is linearly dependent.

Theorem 1.20: A finite set of vectors that contains the zero vector will be linearly dependent.

Proof: Let $S = \{0, v_1, v_2, \dots, v_n\}$ be any set of vectors that contains the zero vector. Consider

$$\alpha_1(0) + \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$$

Which implies $\alpha_1 \neq 0$

Therefore $S = \{0, v_1, v_2, \dots, v_n\}$ linearly dependent.

Theorem 1.21: Let $S = \{v_1, v_2, \dots, v_n\}$ be a linearly independent set of vectors in

a vector space V over a field F . Then every element of $L(S)$ can be uniquely written in the form $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$, where $v_i \in S$ and $\alpha_i \in F$.

Proof: By the definition, every element of $L(S)$ is of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

We prove that every element of $L(S)$ can be uniquely written in the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$$

If not suppose there is linear combination $\beta_1v_1 + \beta_2v_2 + \dots + \beta_nv_n$ of S such that

$$\alpha_1v_1 + \alpha_2v_2 + \dots + \alpha_nv_n = \beta_1v_1 + \beta_2v_2 + \dots + \beta_nv_n, \quad \text{where } \beta_i \in F$$

$$\Rightarrow (\alpha_1 - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + \dots + (\alpha_n - \beta_n)v_n = 0$$

Since S is a linearly independent set, $(\alpha_i - \beta_i) = 0$ for all i .

$$\alpha_i - \beta_i = 0 \text{ for all } i$$

$$\therefore \alpha_i = \beta_i \text{ for all } i$$

Hence every element of $L(S)$ can be uniquely written in the form

$$\alpha_1v_1 + \alpha_2v_2 + \dots + \alpha_nv_n$$

Theorem 1.22: A set $S = \{v_1, v_2, \dots, v_n\}; n \geq 2$ is a linearly dependent set of vectors in V if and only if there exists a vector $v_k \in S$ such that v_k is a linear combination of the preceding vectors v_1, v_2, \dots, v_{k-1} .

1. Determine whether the following sets of vectors $v_3(\mathbb{R})$ are linearly dependent or linearly independent.

- i. $V_1 = (0, 2, -4), V_2 = (1, -2, -1), V_3 = (1, -4, 3)$**
- ii. $V_1 = (1, 2, -3), V_2 = (1, -3, 2), V_3 = (2, -1, 5)$**
- iii. $V_1 = (1, 2, 3), V_2 = (3, 1, 5), V_3 = (3, -4, 7)$**

Solution:

(i) Let $av_1 + bv_2 + cv_3 = 0, a, b, c \in \mathbb{R}$

$$a(0, 2, -4) + b(1, -2, -1) + c(1, -4, 3) = (0, 0, 0)$$

$$\Rightarrow (0, 2a, -4a) + (b, -2b, -b) + (c, -4c, -3c) = (0, 0, 0)$$

$$\Rightarrow (b+c, 2a-2b-4c, -4a-b+3c) = (0, 0, 0)$$

$$b + c = 0 \quad \dots\dots\dots(1)$$

$$2a - 2b - 4c \Rightarrow a - b - 2c = 0 \quad \dots\dots\dots(2)$$

$$- 4a - b + 3c = 0 \quad \dots\dots\dots(3)$$

Subtracting (3) from (2)

$$5a - 5c = 0 \Rightarrow a = c$$

From (1) $b = -c$

If we choose $c = k$, then $a=k$ and $b=-k$

Hence the system is linearly dependent

(ii) $a(1,2,-3)+b(1,-3,2)+c(2,-1,5) = (0,0,0)$

$$a + b + 2c = 0 \quad \dots\dots(1)$$

$$2a - 3b - c = 0 \quad \dots\dots(2)$$

$$-3a + 2b + 5c = 0 \quad \dots\dots\dots(3)$$

Multiply (1) by 2,

$$2a + 2b + 4c = 0 \quad \dots\dots\dots(4)$$

Subtracting (1) and (2),

We get $5b + 5c = 0 \quad \dots\dots\dots(5)$

Multiply (1) by (3),

$$3a + 3b + 6c = 0 \quad \dots\dots\dots(6)$$

Adding (3) and (6),

$$5b = 11c = 0 \quad \dots\dots\dots(7)$$

Substituting $c=0$ in (5)

We get $b=0$

From (1), $a=0$

$$a = 0, b = 0, c = 0$$

The given system is linearly independent.

$$(iii) \quad a(1,2,3) + b(3,1,5) + c(3,-4,7) = (0,0,0)$$

$$a + 3b + 3c = 0 \quad \dots\dots(1)$$

$$2a + b - 4c = 0 \quad \dots\dots(2)$$

$$a + 5b + 7c = 0 \quad \dots\dots(3)$$

Subtracting (3) and (1),

$$2b + 4c = 0 \quad \dots\dots(4)$$

Multiply (1) by (2), $2a + 6b + 6c = 0 \quad \dots(5)$

Subtracting (5) and (2),

$$5b + 10c = 0$$

$$b + 2c = 0 \quad \dots\dots(6)$$

Multiplying (6) by 2,

$$2b + 4c = 0 \quad \dots\dots(7)$$

From (4) and (7),

$$B = -2c$$

Substituting b in (2)

$$2a - 2c - 4c = 0$$

$$2a = 6c$$

$$a = 3c$$

The given system is linearly dependent.

2.If $V_1 = (2, -1, 0)$, $V_2 = (1, 2, 1)$ and $V_3 = (0, 2, -1)$. Show V_1, V_2, V_3 are linearly independent. Is it possible $(3, 2, 1)$ as a linear combination of V_1, V_2, V_3 .

Solution:

Let $av_1 + bv_2 + cv_3 = 0$, $a, b, c \in F$

$$a(2, -1, 0) + b(1, 2, 1) + c(0, 2, -1) = (0, 0, 0)$$

$$2a + b = 0 \quad \dots\dots(1)$$

$$-a + 2b + 2c = 0 \quad \dots\dots(2)$$

$$b - c = 0 \quad \dots\dots(3)$$

these equation can be put in the form $AX = 0$

$$\begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Det } A = \det \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$= \det \begin{bmatrix} 2 & 1 & 0 \\ -1 & 4 & 2 \\ 0 & 0 & -1 \end{bmatrix} \quad C_1 \rightarrow C_2 + C_3$$

$$= -\det \begin{bmatrix} 2 & 1 \\ -1 & 4 \end{bmatrix} = -9 \neq 0$$

$$a = b = c = 0$$

hence the system is linearly independent.

Let $v = a_1v_1 + a_2v_2 + a_3v_3$ where $a_1, a_2, a_3 \in F$

$$(3, 2, 1) = a_1(2, -1, 0) + a_2(1, 2, 1) + a_3(0, 2, -1)$$

$$(3, 2, 1) = (2a_1 + a_2, -a_1 + 2a_2 + 2a_3, a_2 - a_3)$$

Comparing

$$3 = 2a_1 + a_2 \quad \dots\dots\dots(4)$$

$$2 = -a_1 + 2a_2 + a_3 \quad \dots\dots\dots(5)$$

$$1 = a_2 - a_3 \quad \dots\dots\dots(6)$$

Multiplying (5) by 2,

$$4 = 2(-a_1 + 2a_2 + a_3) \quad \dots\dots\dots(7)$$

Adding (4) and (7)

$$7 = 5a_2 + 4a_3 \quad \dots\dots\dots(8)$$

Multiplying (6) by 5,

$$5 = 5a_2 + 5a_3 \quad \dots\dots\dots(9)$$

Subtracting (8) and (9)

$$2 = 9a_3 \Rightarrow a_3 = 2/9$$

Substituting a_3 in (6)

$$1 = a_2 - 2/9 \Rightarrow 1 + 2/9$$

$$a_2 = \frac{11}{9}$$

Substituting a_2 in (4)

$$3 = 2a_1 + \frac{11}{9}$$

$$2a_1 = 3 - \frac{11}{9}$$

$$2a_1 = \frac{27-11}{9}$$

$$\Rightarrow a_1 = \frac{16}{2 \cdot 9} = \frac{8}{9}$$

$$a_1 = \frac{8}{9}, a_2 = \frac{11}{9}, a_3 = \frac{2}{9}$$

$$\text{hence } (3, 2, 1) = \frac{8}{9}(2, -1, 0) + \frac{11}{9}(1, 2, 1) + \frac{2}{9}(0, 2, -1)$$

which is the required linear combination.

1. If x, y, z are linearly independent vectors in a vector space V then prove that all linearly independent $x+y, x-y, x-2y+z$

Solution:

Let $a, b, c \in F$ such that

$$a(x+y) + b(x-y) + c(x-2y+z) = 0$$

$$\Rightarrow (a+b+c)x + (a-b-2c)y + cz = 0, \quad x \neq 0, y \neq 0, z \neq 0$$

$$\text{Comparing } a + b + c = 0 \dots (1)$$

$$a - b - 2c = 0 \dots (2),$$

$$c = 0 \dots (3)$$

Note:

1. Any matrix with distinct eigen values can be diagonalizable.
2. All matrices do not possess n linearly independent eigen vectors. Therefore all matrices are not diagonalizable.
3. Similar matrices have the same eigen values.
4. If A is diagonalizable then it has n linearly independent eigen vectors.
5. Symmetric matrices are always diagonalizable.
6. Let A be a square matrix, A is orthogonally diagonalizable iff it is a symmetric matrix.

Definition:

A square matrix A is said to be orthogonally diagonalizable if there exists an orthogonal matrix N such that $D = N^T A N$ is a diagonal matrix.

1. Show that the following matrix $A = \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix}$ is diagonalizable hence find A^9 .

Solution:

The characteristic equation is given by $|A - \lambda I| = 0$

$$\text{(i.e.,)} \begin{vmatrix} -4 - \lambda & -6 \\ 3 & 5 - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (-4 - \lambda)(5 - \lambda) - 3(-6) = 0$$

$$\Rightarrow -20 + 4\lambda - 5\lambda + \lambda^2 + 18 = 0$$

$$\Rightarrow \lambda^2 - \lambda - 2 = 0$$

$$(\lambda + 1)(\lambda - 2) = 0$$

$$\lambda = -1, 2$$

The eigen values are $\lambda = -1, 2$

To find eigen vectors :

$$(A - \lambda I)v = 0$$

$$\begin{vmatrix} -4 - \lambda & -6 \\ 3 & 5 - \lambda \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \dots\dots(1)$$

Case (i)

Substituting $\lambda = 2$ in we get

$$\begin{vmatrix} -4 - 2 & -6 \\ 3 & 5 - 2 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} -6 & -6 \\ 3 & 3 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-6x_1 - 6x_2 = 0$$

$$3x_1 + 3x_2 = 0 \Rightarrow 3x_1 = -3x_2$$

$$\Rightarrow x_1 = -x_2$$

Let $x_2 = t$, then $x_1 = t$

$$V_1 = t \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Case (ii)

Substituting $\lambda = -1$ in we get

$$\begin{vmatrix} -4 + 1 & -6 \\ 3 & 5 + 1 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{vmatrix} -3 & -6 \\ 3 & 6 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3x_1 - 6x_2 = 0$$

$$3x_1 + 6x_2 = 0 \Rightarrow 3x_1 = -6x_2$$

$$\Rightarrow x_1 = -2x_2$$

Let $x_2 = s$, then $x_1 = -2s$

$$V_2 = s \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Since A has two linearly independent eigen vectors it is diagonalizable.

Modal matrix is the column vectors of the diagonalizing matrix M.

$$M = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$M^{-1} A M = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$M^{-1} = \frac{1}{|M|} (\text{cofactor matrix})^T$$

$$= \frac{1}{(-1+2)} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

Substituting M^{-1} in (2),

$$M^{-1} AM = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -4 & -6 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4+6 & -6+10 \\ 4-3 & 6-5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -2+4 & -4+4 \\ -1+1 & -2+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} = D$$

$$M^{-1} AM = D \dots\dots\dots(3)$$

Pre-multiply (3) by M and postmultiply (3) by M^{-1} on both

$$MM^{-1} AM M^{-1} = MDM^{-1}$$

$$A = MDM^{-1}$$

$$A^9 = MD^9 M^{-1} \dots\dots\dots (4)$$

$$D^9 = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}^9 = \begin{bmatrix} 2^9 & 0 \\ 0 & (-1)^9 \end{bmatrix}$$

$$= \begin{bmatrix} 512 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A^9 = MD^9 M^{-1}$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 512 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 512 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -512 + 0 & 0 + 2 \\ 512 + 0 & 0 - 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -514 & +1026 \\ 513 & 1025 \end{bmatrix}$$

