THE STABLE MARRIAGE PROBLEM.

Stable Marriage Problem

- There is a set Y = {m_1,…,m_n} of *n* men and a set X = {w_1,…,w_n} of *n* women. Each man has a ranking list of the women, and each woman has a ranking list of the men (with no ties in these lists).
- A *marriage matching* M is a set of *n* pairs (m_i,w_j).
- A pair (*m, w*) is said to be a *blocking pair* for matching M if man *m* and woman *w* are not matched in M but prefer each other to their mates in M.
- A marriage matching M is called *stable* if there is no blocking pair for it; otherwise, it's called *unstable*.
- The*stablemarriageproblem*istofindastablemarriagematchingformen'sandwomen's given preferences.

Instance of the Stable Marriage Problem

An instance of the stable marriage problem can be specified either by two sets of preference lists or by a ranking matrix, as in the example below.

men'spreferences

	1st	2nd	3rd
Bob:	Lea	Ann	Sue
Jim:	Lea	Sue	Ann
Tom:	Sue	Lea	Ann

women's preferences

	1st	2nd	3rd
Ann:	Jim	Tom	Bob
Lea:	Tom	Bob	Jim
Sue:	Jim	Tom	Bob

ranking matrix

	Ann	Lea	Sue
Bob	2,3	1,2	3,3
Jim	3,1	1,3	2,1
Tom	3,2	2,1	1,2

Stable Marriage Algorithm (Gale-Shapley)

Step 0: Start with all the men and women being free

Step 1 : While there are free men, arbitrarily select one of them and do the following:

- *Proposal* The selected free man *m* proposes to *w*, the next woman on his preference list
- *Response* If *w* is free, she accepts the proposal to be matched with *m*. If she is not free, she compares *m* with her current mate. If she prefers *m* to him, she accepts *m*'s proposal, making her former mate free; otherwise, she simply rejects *m*'s proposal, leaving *m* free

Step 2 Return the set of *n* matched pairs

Example

Free men: Bob, Jim, Tom

	Ann	Lea	Sue
Bob	2,3	**1,2**	3,3
Jim	3,1	1,3	2,1
Tom	3,2	2,1	1,2

Bob proposed to Lea, Lea accepted Bob Free men: Jim, Tom

	Ann	Lea	Sue
Bob	2,3	1,2	3,3
Jim	3,1	1,3	2,1
Tom	3,2	2,1	1,2

Jim proposed to Lea,

Lea rejected Free men:

Jim, Tom

	Ann	Lea	Sue
Bob	2,3	**1,2**	3,3
Jim	3,1	1,3	**2,1**
Tom	3,2	2,1	1,2

Jim proposed to Sue, Sue

accepted Free men: Tom

	Ann	Lea	Sue
Bob	2,3	**1,2**	3,3
Jim	3,1	1,3	**2,1**
Tom	3,2	2,1	1,2

Tom proposed to Sue,

Sue rejected Free men:

Tom

	Ann	Lea	Sue
Bob	2,3	1,2	3,3
Jim	3,1	1,3	**2,1**
Tom	3,2	**2,1**	1,2

Tom proposed to Lea, Lea replaced Bob

with Tom Free men: Bob

	Ann	Lea	Sue
Bob	2,3	1,2	3,3
Jim	3,1	1,3	2,1
Tom	3,2	2,1	1,2

Bob proposed to Ann, Ann accepted

An accepted proposal is indicated by a boxed cell; a rejected proposal is shown by an underlined cell.

Analysis of the Gale-Shapley Algorithm

- The algorithm terminates after no more than n^2 iterations with a stable marriage output.
- The stable matching produced by the algorithm is always *man-optimal*: each man gets the highest rank woman on his list under any stable marriage. One can obtain the *woman- optimal* matching by making women propose to men.
- A man (woman) optimal matching is unique for a given set of participant preferences.
- The stable marriage problem has practical applications such as matching medical-school graduates with hospitals for residency training.