Electrostatics

In this chapter we will discuss on the followings:

1. Coulomb's Law
2. Electric Field & Electric Flux Density
3. Gauss's Law with Application
4. Electrostatic Potential, Equipotential Surfaces
5. Boundary Conditions for Static Electric Fields
6. Capacitance and Capacitors
7. Electrostatic Energy
8. Laplace's and Poisson's Equations
9. Uniqueness of Electrostatic Solutions
10. Method of Images
11. Solution of Boundary Value Problems in Different Coordinate Systems

Introduction

In the previous chapter we have covered the essential mathematical tools needed to study EM fields. We have already mentioned in the previous chapter that electric charge is a fundamental property of matter and charge exist in integral multiple of electronic charge. Electrostatics can be defined as the study of electric charges at rest. Electric fields have their sources in electric charges.

(Note: Almost all real electric fields vary to some extent with time. However, for many problems, the field variation is slow and the field may be considered as static. For some other cases spatial distribution is nearly same as for the static case even though the actual field may vary with time. Such cases are termed as quasi-static.)

In this chapter we first study two fundamental laws governing the electrostatic fields, viz, (1) Coulomb's Law and (2) Gauss's Law. Both these law have experimental basis. Coulomb's law is applicable in finding electric field due to any charge distribution, Gauss's law is easier to use when the distribution is symmetrical.

Coulomb's Law

Coulomb's Law states that the force between two point charges $Q1$ and $Q2$ is directly proportional to the product of the charges and inversely proportional to the square of the distance between them.

Point charge is a hypothetical charge located at a single point in space. It is an idealised model of a particle having an electric charge.

Mathematically, $F = \frac{kQ_1Q_2}{R^2}$,where k is the proportionality constant.

In SI units, Q_1 and Q_2 are expressed in Coulombs(C) and R is in meters.

Force F is in Newtons (N) and $k = \frac{1}{4\pi\varepsilon_0}$, ε_0 is called the permittivity of free space.

(We are assuming the charges are in free space. If the charges are any other dielectric medium, we will use $\varepsilon = \varepsilon_0 \varepsilon_r$ instead where ε_r is called the relative permittivity or the dielectric constant of the medium).

Therefore $F = \frac{1}{4\pi\varepsilon_0}\frac{Q_1Q_2}{R^2}$ ....(2.1)

As shown in the Figure 2.1 let the position vectors of the point charges $Q_1$and Q_2 are given by $\vec{r_1}$ and $\vec{r_2}$. Let $\vec{F_{12}}$ represent the force on Q_1 due to charge Q_2.

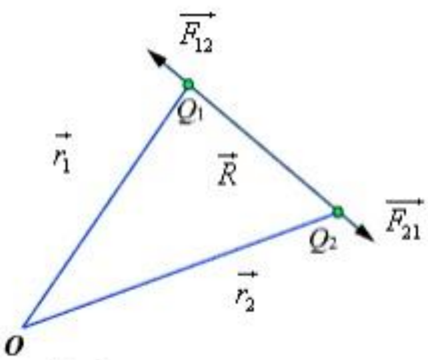

Fig 2.1: Coulomb's Law

The charges are separated by a distance of $R = \left|\vec{r_1} - \vec{r_2}\right| = \left|\vec{r_2} - \vec{r_1}\right|$. We define the unit vectors as

$\widehat{a_{12}} = \frac{(\vec{r_2} - \vec{r_1})}{R}$ and $\widehat{a_{21}} = \frac{(\vec{r_1} - \vec{r_2})}{R}$ ....(2.2)

$\overrightarrow{F_{12}}$ can be defined as $\overrightarrow{F_{12}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\widehat{a_{12}} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\frac{(\vec{r_2} - \vec{r_1})}{|\vec{r_2} - \vec{r_1}|^3}$. Similarly the force on Q_1 due to charge Q_2 can be calculated and if $\overrightarrow{F_{21}}$ represents this force then we can write $\overrightarrow{F_{21}} = -\overrightarrow{F_{12}}$

When we have a number of point charges, to determine the force on a particular charge due to all other charges, we apply principle of superposition. If we have N number of charges $Q_1, Q_2, ...Q_N$ located respectively at the points represented by the position vectors $\vec{r_1}, \vec{r_2}, ...\vec{r_N}$, the force experienced by a charge Q located at $\vec{r}$ is given by,

$$\vec{F} = \frac{Q}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{Q_i(\vec{r} - \vec{r_i})}{|\vec{r} - \vec{r_i}|^3}$$ ....(2.3)

Electric Field

The electric field intensity or the electric field strength at a point is defined as the force per unit charge. That is

$$\vec{E} = \lim_{Q \to 0} \frac{\vec{F}}{Q} \text{ or, } \quad \vec{E} = \frac{\vec{F}}{Q} \quad \text{.......................................(2.4)}$$

The electric field intensity *E* at a point *r* (observation point) due a point charge *Q* located at $\vec{r}'$ (source point) is given by:

$$\vec{E} = \frac{Q(\vec{r} - \vec{r}')}{4\pi\varepsilon_0 \left|\vec{r} - \vec{r}'\right|^3} \quad \text{..(2.5)}$$

For a collection of *N* point charges $Q_1, Q_2, \ldots\ldots\ldots Q_N$ located at $\vec{r_1}, \vec{r_2}, \ldots\ldots \vec{r_N}$, the electric field intensity at point $\vec{r}'$ is obtained as

$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q_k(\vec{r} - \vec{r_i})}{\left|\vec{r} - \vec{r_i}\right|^3} \quad \text{...(2.6)}$$

The expression (2.6) can be modified suitably to compute the electric filed due to a continuous distribution of charges.

In figure 2.2 we consider a continuous volume distribution of charge at t in the region denoted as the source region.

For an elementary charge $dQ = \rho(\vec{r'})dv'$, i.e. considering this charge as point charge, we can write the field expression as:

$$d\vec{E} = \frac{dQ(\vec{r} - \vec{r'})}{4\pi\varepsilon_0 |\vec{r} - \vec{r'}|^3} = \frac{\rho(\vec{r'})dv'(\vec{r} - \vec{r'})}{4\pi\varepsilon_0 |\vec{r} - \vec{r'}|^3} \quad \text{.............(2.7)}$$

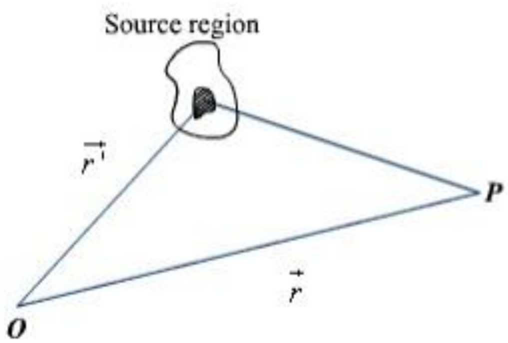

Fig 2.2: Continuous Volume Distribution of Charge

When this expression is integrated over the source region, we get the electric field at the point *P* due to this distribution of charges. Thus the expression for the electric field at *P* can be written as:

$$\overline{E(r)} = \int_V \frac{\rho(\vec{r'})(\vec{r} - \vec{r'})}{4\pi\varepsilon_0 |\vec{r} - \vec{r'}|^3} dv' \quad \text{...(2.8)}$$

Similar technique can be adopted when the charge distribution is in the form of a line charge density or a surface charge density.

$$\overline{E(r)} = \int_L \frac{\rho_L(\vec{r'})(\vec{r} - \vec{r'})}{4\pi\varepsilon_0 |\vec{r} - \vec{r'}|^3} dl' \quad \text{..(2.9)}$$

$$\overline{E(r)} = \int_S \frac{\rho_s(\vec{r'})(\vec{r} - \vec{r'})}{4\pi\varepsilon_0 |\vec{r} - \vec{r'}|^3} ds' \quad \text{..(2.10)}$$