5.4 FORMATION OF DIFFERENCE EQUATION & SOLUTION OF DIFFERENCE EQUATION

	Derive the difference equation from $u_n = a + b3^n$ **Solution:** $u_n = a + b3^n \quad ----(1)$ Replace n by $n+1$ in (1) $u_{n+1} = a + b3^{n+1}$ $u_{n+1} = a + 3b3^n \quad ----(2)$ Replace n by $n+2$ in (1) $u_{n+2} = a + b3^{n+2}$ $u_{n+2} = a + 9b3^n \quad ----(3)$ From (1), (2) and (3) $\begin{vmatrix} u_n & 1 & 1 \\ u_{n+1} & 1 & 3 \\ u_{n+2} & 1 & 9 \end{vmatrix} = 0$ $u_n(9-3) - 1(3u_{n+2} - 9u_{n+1}) + 1(u_{n+1} - u_{n+2}) = 0$ $6u_n - 3u_{n+2} + 9u_{n+1}) + u_{n+1} - u_{n+2} = 0$ $-4u_{n+2} + 10u_{n+1} + 6u_n = 0$ $\div(-2) \quad \Rightarrow \boxed{2u_{n+2} - 5u_{n+1} - 3u_n = 0}$
3.	**Form the difference equation** $y_n = \cos\left(\frac{n\pi}{2}\right)$ **Solution:** Given $y_n = \cos\left(\frac{n\pi}{2}\right) \quad ----(1)$ Replace n by $n+1$ in (1) $y_{n+1} = \cos\left(\frac{(n+1)\pi}{2}\right) = \cos\left(\frac{\pi}{2} + \frac{n\pi}{2}\right) = -\sin\left(\frac{n\pi}{2}\right) \quad ----(2)$ Replace n by $n+2$ in (1) $y_{n+2} = \cos\left(\frac{(n+2)\pi}{2}\right) = \cos\left(\frac{2\pi}{2} + \frac{n\pi}{2}\right)$ $y_{n+2} = \cos\left(\pi + \frac{n\pi}{2}\right) = -\cos\left(\frac{n\pi}{2}\right)$ $y_{n+2} = -y_n \quad$ from (1) $\Rightarrow \boxed{y_{n+2} + y_n = 0}$
Solutions of difference equation using Z-Transforms. 1. $Z[y_n] = Z[y(n)] = y(z)$	

2. $Z[y_{n+1}] = Z[y(n+1)] = zy(z) - zy(0)$ 3. $Z[y_{n+2}] = Z[y(n+2)] = z^2y(z) - z^2y(0) - zy(1)$ 4. $Z[y_{n+3}] = Z[y(n+3)] = z^3y(z) - z^3y(0) - z^2y(1) - zy(2)$	

1. **Solve using Z-transforms technique the difference equation** $y_{n+2} + 4y_{n+1} + 3y_n = 3^n$ **with** $y_0 = 0, y_1 = 1$.

Solution:

$$y_{n+2} + 4y_{n+1} + 3y_n = 3^n \; .$$

Taking Z-transform on both sides

$$Z[y_{n+2}] + 4Z[y_{n+1}] + 3Z[y_n] = Z[3^n]$$

$$\left[z^2y(z) - z^2y(0) - zy(1)\right] + 4[zy(z) - zy(0)] + 3y(z) = \frac{z}{z-3}$$

Given $y_0 = y(0) = 0, y_1 = y(1) = 1$

$$z^2y(z) - z + 4zy(z) + 3y(z) = \frac{z}{z-3}$$

$$(z^2 + 4z + 3)y(z) = \frac{z}{z-3} + z$$

$$(z^2 + 4z + 3)y(z) = \frac{z + z^2 - 3z}{z-3}$$

$$y(z) = \frac{z^2 - 2z}{(z-3)(z^2 + 4z + 3)}$$

$$y(z) = \frac{z(z-2)}{(z-3)(z+1)(z+3)}$$

By Partial Fraction,

$$\frac{y(z)}{z} = \frac{(z-2)}{(z-3)(z+1)(z+3)} \quad - - - - - (1)$$

Now $\dfrac{(z-2)}{(z-3)(z+1)(z+3)} = \dfrac{A}{(z-3)} + \dfrac{B}{(z+1)} + \dfrac{C}{(z+3)}$

$$z - 2 = A(z+1)(z+3) + B(z-3)(z+3) + C(z+1)(z-3)$$

<u>Put z=3</u> $\Rightarrow 1 = 24A \Rightarrow A = \dfrac{1}{24}$

<u>Put z= -1</u> $\Rightarrow -3 = -8B \Rightarrow B = \dfrac{3}{8}$

<u>Put z = -3</u> $\Rightarrow -5 = 12C \Rightarrow C = \dfrac{-5}{12}$

$$\frac{(z-2)}{(z-3)(z+1)(z+3)} = \frac{1/24}{(z-3)} + \frac{3/8}{(z+1)} + \frac{-5/12}{(z+3)}$$

$$(1) \Rightarrow \frac{y(z)}{z} = \frac{1/24}{(z-3)} + \frac{3/8}{(z+1)} + \frac{-5/12}{(z+3)}$$

$$y(z) = \frac{1}{24}\frac{z}{(z-3)} + \frac{3}{8}\frac{z}{(z+1)} - \frac{5}{12}\frac{z}{(z+3)}$$

Taking Z^{-1} on both sides

$$Z^{-1}[y(z)] = \frac{1}{24}Z^{-1}\left[\frac{z}{z-3}\right] + \frac{3}{8}Z^{-1}\left[\frac{z}{z+1}\right] - \frac{5}{12}Z^{-1}\left[\frac{z}{z+3}\right]$$

	$y(\mathrm{n}) = \frac{1}{24}(3)^n + \frac{3}{8}(-1)^n - \frac{5}{12}(-3)^n$ $\quad \because Z^{-1}\left[\frac{z}{z-a}\right] = a^n$
2.	**Solve $y_{n+2} - 3y_{n+1} - 10\, y_n = 0$, given $y_0 = 1$, $y_1 = 0$.** **Solution:** $y_{n+2} - 3y_{n+1} - 10y_n = 0$. Taking Z-transform on both sides $Z[y_{n+2}] - 3Z[y_{n+1}] - 10Z[y_n] = Z[0]$ $[z^2 y(z) - z^2 y(0) - zy(1)] - 3[zy(z) - zy(0)] - 10y(z) = 0$ Given $y_0 = y(0) = 1, y_1 = y(1) = 0$ $z^2 y(z) - z^2 - 3zy(z) + 3z - 10y(z) = 0$ $(z^2 - 3z - 10)y(z) = z^2 - 3z$ $y(z) = \frac{z^2 - 3z}{(z^2 - 3z - 10)}$ $y(z) = \frac{z(z-3)}{(z+2)(z-5)}$ By Partial Fraction, $\frac{y(z)}{z} = \frac{(z-3)}{(z+2)(z-5)} \quad -----(1)$ Now $\frac{(z-3)}{(z+2)(z-5)} = \frac{A}{(z+2)} + \frac{B}{(z-5)}$ $z - 3 = A(z-5) + B(z+2)$ <u>Put z= - 2</u> $\Rightarrow -5 = -7A \Rightarrow A = \frac{5}{7}$ <u>Put z= 5</u> $\Rightarrow 2 = 7B \Rightarrow B = \frac{2}{7}$ $\frac{(z-3)}{(z+2)(z-5)} = \frac{\frac{5}{7}}{(z+2)} + \frac{\frac{2}{7}}{(z-5)}$ $(1) \Rightarrow \frac{y(z)}{z} = \frac{\frac{5}{7}}{z+2} + \frac{\frac{2}{7}}{z-5}$ $y(z) = \frac{5}{7}\frac{z}{z+2} + \frac{2}{7}\frac{z}{z-5}$ Taking Z^{-1} on both sides $Z^{-1}[y(z)] = \frac{5}{7}Z^{-1}\left[\frac{z}{z+2}\right] + \frac{2}{7}Z^{-1}\left[\frac{z}{z-5}\right]$ $y(\mathrm{n}) = \frac{5}{7}(-2)^n - \frac{2}{7}5^n$ $\quad \because Z^{-1}\left[\frac{z}{z-a}\right] = a^n$
3.	**Solve the equation $y(n+3) - 3y(n+1) + 2y(n) = 0$ given that $y(0) = 4$, $y(1) = 0$ and $y(2) = 8$.** **Solution:** $Z[y(n+3)] - 3Z[y(n+1)] + 2Z[y(n)] = Z[0]$ $[z^3 y(z) - z^3 y(0) - z^2 y(1) - zy(2)] - 3[zy(z) - zy(0)] + 2y(z) = 0$ Given that $y(0) = 4$, $y(1) = 0$

	$z^3y(z)-4z^3-8z-3zy(z)+12z+2y(z)=0$ $\left[z^3-3z+2\right]y(z)=4z^3-4z$ $y(z)=\dfrac{4z^3-4z}{z^3-3z+2}$ $y(z)=\dfrac{4z(z^2-1)}{(z-1)^2(z+2)}$ $y(z)=\dfrac{4z\,(z-1)\,(z+1)}{(z-1)^2\,(z+2)}$ $\quad\because a^2-b^2=(a+b)(a-b)$ $y(z)=\dfrac{4z(z+1)}{(z-1)(z+2)}$ By Partial Fraction $\dfrac{y(z)}{z}=\dfrac{4(z+1)}{(z-1)(z+2)}\quad -----(1)$ $\dfrac{4(z+1)}{(z-1)(z+2)}=\dfrac{A}{z-1}+\dfrac{B}{z+2}$ $4(z+1)=A(z+2)+B(z-1)$ Put $z=1\Rightarrow 8=3A\Rightarrow A=\dfrac{8}{3}$ Put $z=-2\Rightarrow -4=-3B\Rightarrow B=\dfrac{4}{3}$ $\dfrac{y(z)}{z}=\dfrac{8/3}{z-1}+\dfrac{4/3}{z+2}$ $Z^{-1}\left[y(z)\right]=\dfrac{8}{3}Z^{-1}\left[\dfrac{z}{z-1}\right]+\dfrac{4}{3}Z^{-1}\left[\dfrac{z}{z+2}\right]$ $\boxed{y(n)=\dfrac{8}{3}+\dfrac{4}{3}(-2)^n}$ $\quad\because Z^{-1}\left[\dfrac{z}{z-a}\right]=a^n$
4.	**Using Z-transform solve** $y(n)+3y(n-1)-4y(n-2)=0, n\geq 2$ **given that** $y(0)=3$ and $y(1)=-2$ **Solution:** **Given** $y(n)+3y(n-1)-4y(n-2)=0, n\geq 2$ **Replace** n by $n+2$, we get $y(n+2)+3y(n+1)-4y(n)=0$ Taking Z transforms on both sides $Z\left[y(n+2)\right]+3Z\left[y(n+1)\right]-4Z\left[y(n)\right]=Z\left[0\right]$ $\left[z^2y(z)-z^2y(0)-zy(1)\right]+3\left[zy(z)-zy(0)\right]-4y(z)=0$ Given that $y(0)=3$ and $y(1)=-2$ $\left[z^2y(z)-3z^2+2z\right]+3\left[zy(z)-3z\right]-4y(z)=0$ $\left[z^2+3z-4\right]y(z)-3z^2+2z-9z=0$ $\left[z^2+3z-4\right]y(z)=3z^2+7z$ $y(z)=\dfrac{3z^2+7z}{z^2+3z-4}$ By Partial Fraction $\dfrac{y(z)}{z}=\dfrac{3z+7}{z^2+3z-4}=\dfrac{3z+7}{(z+4)(z-1)}$

	Now, $\frac{3z+7}{(z+4)(z-1)} = \frac{A}{z+4} + \frac{B}{z-1}$ $3z+7 = A(z-1)+B(z+4)$ Put $z=1 \Rightarrow 10=5B \Rightarrow B=2$ Put $z=-4 \Rightarrow -5=-5A \Rightarrow A=1$ $\frac{y(z)}{z} = \frac{1}{z+4} + \frac{2}{z-1}$ $y(z) = \frac{z}{z+4} + 2\frac{z}{z-1}$ $Z^{-1}\left[y(z)\right] = Z^{-1}\left[\frac{z}{z+4}\right] + 2Z^{-1}\left[\frac{z}{z-1}\right]$ $\boxed{y(n) = (-4)^n + 2(1)^n = 2+(-4)^n}$ $\quad\because Z^{-1}\left[\frac{z}{z-a}\right] = a^n$
5.	**Solve using Z-transforms technique the difference equation** $u_{n+2}+6u_{n+1}+9u_n = 2^n$ **with** $u_0 = u_1 = 0$. Solution: $u_{n+2}+6u_{n+1}+9u_n = 2^n$ Assume u=y $y_{n+2}+6y_{n+1}+9y_n = 2^n \quad ; y_0 = y_1 = 0$ Taking Z-transform on both sides $Z\left[y_{n+2}\right]+6Z\left[y_{n+1}\right]+9Z\left[y_n\right] = Z\left[2^n\right]$ $\left[z^2y(z)-z^2y(0)-zy(1)\right]+6\left[zy(z)-zy(0)\right]+9y(z) = \frac{z}{z-2}$ Given $y_0 = y(0) = 0$; $y_1 = y(1) = 0$ $z^2y(z)+6zy(z)+9y(z) = \frac{z}{z-2}$ $(z^2+6z+9)y(z) = \frac{z}{z-2}$ $y(z) = \frac{z}{(z-2)(z^2+6z+9)}$ $y(z) = \frac{z}{(z-2)(z+3)^2}$ By Partial Fraction, $\frac{y(z)}{z} = \frac{1}{(z-2)(z+3)^2} \quad -----(1)$ Now $\frac{1}{(z-2)(z+3)^2} = \frac{A}{(z-2)} + \frac{B}{(z+3)} + \frac{C}{(z+3)^2}$ $1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2)$ Put $z=2 \Rightarrow 1=25A \Rightarrow A=\frac{1}{25}$ Put $z=-3 \Rightarrow 1=-5C \Rightarrow C=\frac{-1}{5}$ Equating co-effT. of z^2 on both sides $\Rightarrow A+B=0 \Rightarrow B=-A \Rightarrow B=-\frac{1}{25}$

	$\frac{y(z)}{z} = \frac{\frac{1}{25}}{(z-2)} + \frac{\frac{-1}{25}}{(z+3)} + \frac{\frac{-1}{5}}{(z+3)^2}$ Taking Z^{-1} on both sides $Z^{-1}[y(z)] = \frac{1}{25}Z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25}Z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{5}Z^{-1}\left[\frac{z}{(z+3)^2}\right]$ $\boxed{y(\mathrm{n}) = \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n - \frac{1}{5}n(-3)^{n-1}}$ $\because Z^{-1}\left[\frac{z}{(z-a)^2}\right] = na^{n-1}$ & $Z^{-1}\left[\frac{z}{z-a}\right] = a^n$ $\boxed{u(\mathrm{n}) = \frac{1}{25}(2)^n - \frac{1}{25}(-3)^n - \frac{1}{5}n(-3)^{n-1}}$ $\because u = y$
6.	**Using Z-transform method solve** $y(k+2)+y(k)=2$ **given that** $y_0 = y_1 = 0$. **Solution:** Given $y(k+2)+y(k)=2$; $y_0 = y_1 = 0$. Assume k=n $y(n+2)+y(n)=2$ Taking Z-transform on both sides $Z[y(n+2)]+Z[y(n)]=2Z[1]$ $[z^2y(z)-z^2y(0)-zy(1)]+y(z)=2\frac{z}{z-1}$ Given that $y_0 = y_1 = 0$. $(z^2+1)y(z)=\frac{2z}{z-1}$ $y(z)=\frac{2z}{(z-1)(z^2+1)}$ $\frac{y(z)}{z}=\frac{2}{(z-1)(z^2+1)}$ -----(1) By partial fraction Now, $\frac{2}{(z-1)(z^2+1)} = \frac{A}{z-1}+\frac{B}{z^2+1}+\frac{Cz}{z^2+1}$ $2 = A(z^2+1)+B(z-1)+Cz(z-1)$ Put $z=1 \Rightarrow 2=2A \Rightarrow A=1$ Put $z=0 \Rightarrow 2=A-B \Rightarrow B=A-2 \Rightarrow B=-1$ Equating co-efft. of z^2 on both sides $\Rightarrow 0=A+C \Rightarrow C=-A \Rightarrow C=-1$ $(1) \Rightarrow \frac{y(z)}{z}=\frac{1}{z-1}+\frac{-1}{z^2+1}+\frac{-z}{z^2+1}$ $y(z)=\frac{z}{z-1}-\frac{z}{z^2+1}-\frac{z^2}{z^2+1}$ Taking Z^{-1} on both sides $Z^{-1}[y(z)]=Z^{-1}\left[\frac{z}{z-1}\right]-Z^{-1}\left[\frac{z}{z^2+1}\right]-Z^{-1}\left[\frac{z^2}{z^2+1}\right]$ $y(n)=(1)^n-1^n\sin\frac{n\pi}{2}-1^n\cos\frac{n\pi}{2}$ $\boxed{y(n)=1-\sin\frac{n\pi}{2}-\cos\frac{n\pi}{2}}$ $\boxed{y(k)=1-\sin\frac{k\pi}{2}-\cos\frac{k\pi}{2}}$