4.5.CONJUGATEBEAMMETHOD

Conjugatebeamisan imaginarybeamoflength equal to thatof theoriginal beam but for which the load diagram is the $\frac{M}{EI}$ diaagram(i.e., the load at any pint on the conjugate beam is equal to the B.M at that point divided by EI).

1. Theslopeat anysection ofthegivenbeamis equalto theshearforce at the corresponding section of the conjugate beam.
2. The deflection at any sectionfor the given beam is equal to the bending moment at the corresponding section of the conjugate beam.

Hence before applying the conjugate beam method, conjugate beam is constructed. Theloadontheconjugatebeamatanypointisequaltothe B.Matthatpointdividedby EI.Hence the loading on the conjugate beam is known.Then the shear force at any point on the conjugate beam gives the slope at the corresponding point of actual beam. And the B.M at any point on the conjugate beam gives the deflection at the corresponding point of the actual beam.

4.5.1 SLOPE AND DEFLECTION OF A SIMPLY SUPPORTED BEAMWITH A POINT LOAD AT CENTRE

Fig. shows a simply supported beam AB of length L carrying a point load W at the centre C.

Sincethebeamissymmetricallyloaded,

$$R_A = R_B = \frac{Total\ load}{2} = \frac{W}{2}$$

BM at the ends A and B= 0 (sinceAandBaresimplysupportedends)

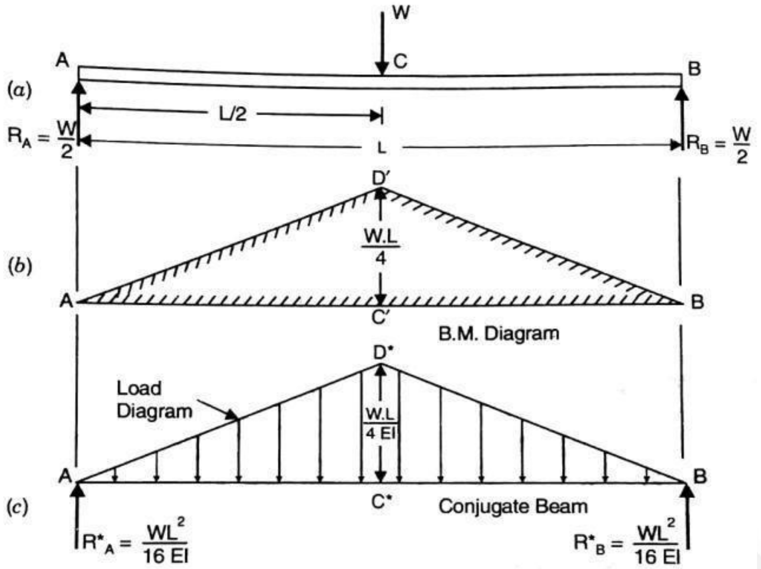
BM at Centre, $C = R_A.\frac{L}{2} = \frac{W}{2}.\frac{L}{2} = \frac{WL}{4}$

TheB.M.diagramisshowninFig.

Now the conjugate beam AB can be constructed.The load on the conjugate beam will be obtained by dividing the B.M at that point by EI.

Theshapeoftheloadingontheconjugatebeamwillbesameasof B.Mdiagram.

The ordinateof loadingonconjugate beam will be equalto $\frac{M}{EI} = \frac{\frac{WL}{4}}{EI} = \frac{WL}{4EI}$



Let R_A*=ReactionatAforconjugatebeam

R_B* = Reaction at B for conjugatebeam

Totalloadontheconjugatebeam

=Areaoftheloaddiagram $= \frac{1}{2} \times AB \times C^*D^* = \frac{1}{2} \times L \times \frac{WL}{4EI}$

$= \frac{WL^2}{8EI}$

Reactionateachsupportfortheconjugatebeamwillbehalfofthetotalload

$R_A^* = \frac{1}{2} \times \frac{WL}{8EI} = \frac{WL}{16EI}$ 2 2 $R_B^*=$

AccordingtoConjugatebeammethod,

Slopeatsupports, θ_A=ShearforceatAfortheconjugatebeam=R_A*

$= \frac{\boldsymbol{WL^2}}{\boldsymbol{16EI}} = \theta_B$

And y_c=B.Mat Cfortheconjugate beam

$= R_A^* \times \frac{L}{2}$-Loadcorrespondingto AC*D*XDistanceof C.G.ofAC*D*fromC

$= \frac{WL^2}{16EI} \times \frac{L}{2} - \left(\frac{1}{2} \times \frac{L}{2} \times \frac{WL}{4EI}\right) X \left(\frac{1}{3} \times \frac{L}{2}\right)$

$= \frac{WL^3}{32EI} - \frac{WL^3}{96EI}$

$\boldsymbol{= \frac{WL^3}{48EI}}$

4.5.2.SLOPE AND DEFLECTION OF A SIMPLY SUPPORTED BEAM WITHUNIFORMLY DISTRIBUTED LOAD

Fig. shows a simply supported beam AB of length L carrying a UDL w/m over its entire length

Sincethebeamissymmetricallyloaded,

$$R_A = R_B = \frac{Total\ load}{2} = \frac{wL}{2}$$

BM at the ends A and B= 0 (sinceAandBaresimplysupportedends)

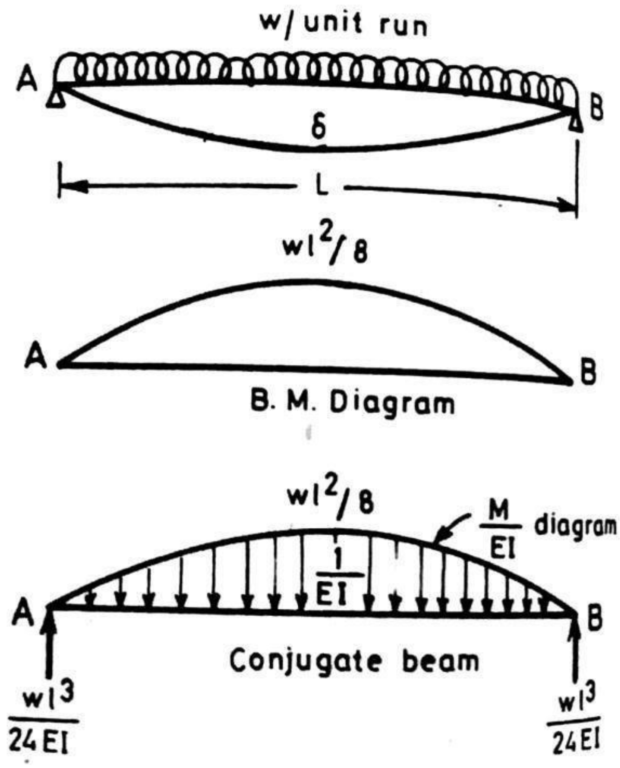
BM at Centre, $C = \frac{WL^2}{8}$

TheB.M.diagramisshowninFig.

NowtheconjugatebeamABcanbeconstructed.Theloadontheconjugatebeam will be obtained by dividing the B.M at that point by EI.

Theshapeoftheloadingontheconjugatebeamwillbesameasof B.Mdiagram.

Theordinateofloadingonconjugatebeamwill beequalto $\frac{M}{EI} = \frac{\frac{WL^2}{8}}{EI} = \frac{WL^2}{8EI}$



Let R_A^*=ReactionatAforconjugate beam

R_B^*=ReactionatBforconjugatebeam Total load on the conjugate beam

=Areaoftheload diagram

$$= \frac{2}{3} X\ L\ X\ \frac{WL^2}{8EI}$$

$$= \frac{WL^3}{12EI}$$

Reaction at each support for the conjugate beam will be half of the total load

$$R_A^* = R_B^* = \frac{1}{2} \times \frac{WL^3}{12EI}$$

$$= \frac{WL^3}{24EI}$$

AccordingtoConjugatebeammethod,

Slopeatsupports, θ_A=ShearforceatAfortheconjugatebeam=R_A^*

$$= \frac{WL^3}{24EI} = \theta_B$$

And y_c=B.Mat Cfortheconjugate beam

$=R_A * \times \frac{L}{2}$-LoadcorrespondingtoAC*D*XDistanceof C.G.ofAC*D*fromC

$$= \frac{WL^3}{24EI} \times \frac{L}{2} - \left(\frac{2}{3} \times \frac{L}{2} \times \frac{WL^2}{8EI}\right) \mathrm{X} \left(\frac{3}{8} \times \frac{L}{2}\right)$$

$$= \frac{WL^4}{48EI} - \frac{6WL^4}{768EI} = \boldsymbol{\frac{5WL^4}{384EI}}$$

Example.4.5.1. AbeamABCDissimplysupportedatit'sAandDoveraspanof30m. It is made up of three portions AB, BC and CD each 10 metres in length.The moments of inertia of the section of these portions are I, 3I and 2I respectively.Where I = 2 X 10^{10}mm^4.Thebeamcarriesapointloadof150KNatBandapointloadof300KNat C.NeglectingtheweightofthebeamcalculatetheslopesanddeflectionsatA,B,Cand D.TakeE=200 kN/mm^2.

Solution:

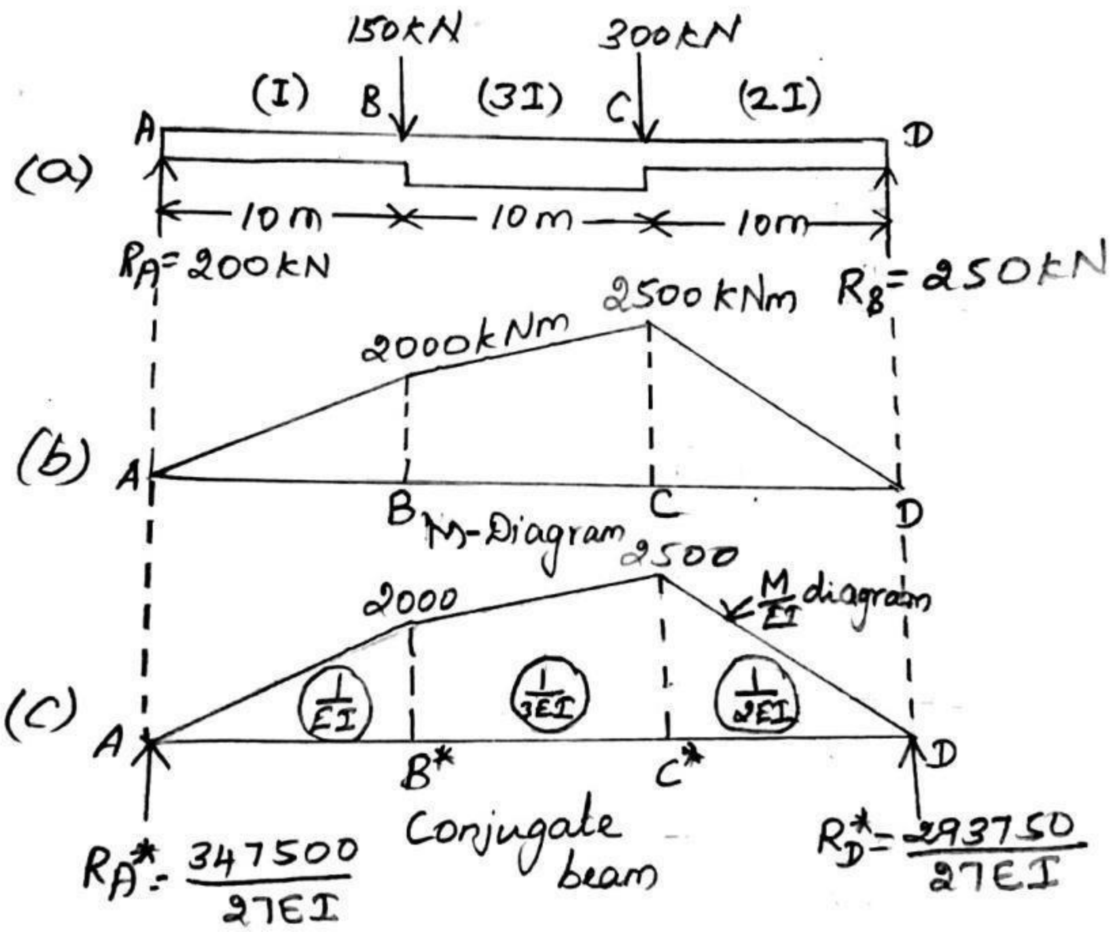
Let$R_A$and$R_B$bethereactionsatthesupports.TakingmomentaboutD,Wehave, $R_{DX}30$ – 150 X 10 -300 X 20 = 0

R_D X30=150X10+300X20 R_D= 250 kN

R_A+R_D=150 +300

R_A+ 250=450

R_A=450-250=200kN.



B.MatA=0

B.Mat B=200 X10=2000kNm

B.MatC =200X 20–150X 10 =2500 kNm.

B.MatD =0

Fig.(b)showstheB.Mdiagramforthegivenbeam.

Fig(c)showsthe $\frac{M}{EI}$ diagramwhichistheloadingontheconjugatebeam.Thethickness onthediagramis $\frac{1}{EI}$ fortheportionAB, $\frac{1}{3EI}$ fortheportionBCand $\frac{1}{2EI}$ fortheportionCD. The properties of the load on the conjugate beam are given below:

Loadcomponent	Magnitude	Distance FromA	Moment AboutA
Load on AB*=$\frac{1}{2}$X 2000 X$\frac{1}{EI}$	$\frac{10000}{EI}$	$\frac{20}{3}$	$\frac{200000}{3EI}$

Load on BC*=2000 X 10 X 500 X $\frac{1}{3EI}$ $\frac{1}{2}$X 10 X500 X $\frac{1}{3EI}$	$\frac{20000}{3EI}$	15	$\frac{100000}{EI}$
	$\frac{2500}{3EI}$	$\frac{50}{3}$	$\frac{125000}{9EI}$
Load on CD* =$\frac{1}{2}$X 10 X 2500X$\frac{1}{2EI}$	$\frac{6250}{EI}$	$\frac{70}{3}$	$\frac{437500}{3EI}$
Total	$\frac{71250}{3EI}$		$\frac{2937500}{9EI}$

Let R_A*and R_D*be the reactions at A and D for the conjugate beam. Taking moments about A, we have,

$R_D^* X 30 = \frac{2937500}{9EI}$

$\therefore R_D^* = \frac{2937500}{270EI} = \frac{293750}{27EI}$

$R_A^* = \frac{71250}{3EI} - \frac{293750}{27EI} = \frac{347500}{27EI}$

Now we can easily determine the slopes and deflections at A,B,C,D for the given beam Slope at A for the given beam = S.F. at A for the conjugate beam

$= \frac{347500}{27EI} = \frac{347500 \, X \, 10^9}{27 \, X \, 200 \, X \, 10^3 X \, 2 \, X \, 10^{10}}$

=**0.003218radians**

Slope at B for the given beam=S.F.at B for the conjugate beam

$= \frac{347500}{27EI} - \frac{1000}{EI} = \frac{77500}{27EI}$

$= \frac{77500 \, X \, 10^9}{27 \, X \, 200 \, X \, 10^3 X \, 2 \, X \, 10^{10}}$=**0.0007176radians**

Slope at C for the given beam=S.F.at C for the conjugate beam

$= \frac{293750}{27EI} - \frac{6250}{EI} = \frac{125000}{27EI}$

$= \frac{125000 \, X \, 10^9}{27 \, X \, 200 \, X \, 10^3 X \, 2 \, X \, 10^{10}}$ =**0.001157radians**

Slope at D for the given beam=S.F.at D for the conjugate beam

$= \frac{293750}{27EI}$

$= \frac{293750 \, X \, 10^9}{27 \, X \, 200 \, X \, 10^3 X \, 2 \, X \, 10^{10}}$=**0.00272 radians**

Deflection at A=B.M at A for the conjugate beam=0

(SinceAissimply supported)

DeflectionatB=B.MatBforconjugatebeam

$$= \frac{347500}{27EI} \times 10 - \frac{10000}{EI} \times \frac{10}{2} = \frac{2575000}{27EI}$$

$$= \frac{2575000 \times 10^{12}}{27 \ X\ 200\ X\ 10^3 X\ 2\ X\ 10^{10}} = \mathbf{23.84mm}$$

DeflectionatC=B.MatCforconjugatebeam

$$= \frac{293750}{27EI} \times 10 - \frac{6250}{EI} \times \frac{10}{3}$$

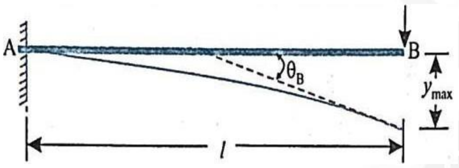
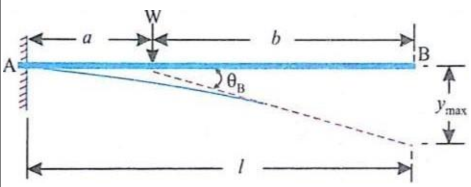
$$= \frac{2375000}{27EI}$$

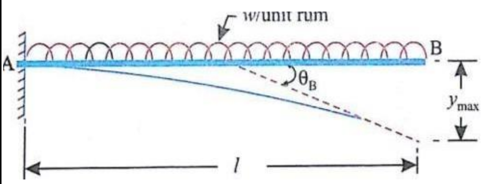
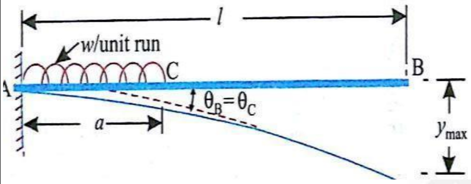
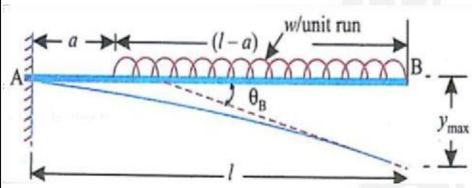
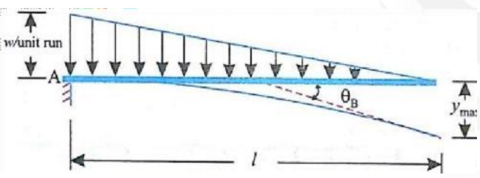
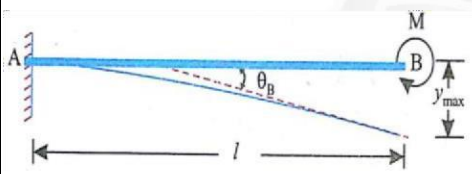
$$= \frac{2375000\ X\ 10^{12}}{27\ X\ 200\ X\ 10^3 X\ 2\ X\ 10^{10}} = \mathbf{21.99mm}$$

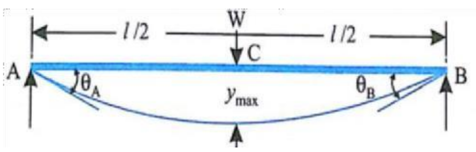
DeflectionatD=B.MatDfortheconjugatebeam=0

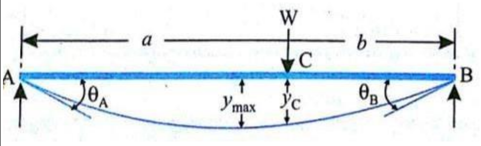
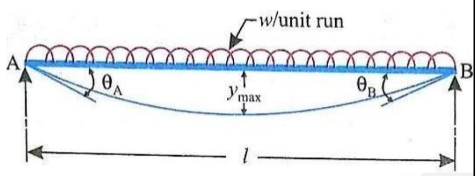
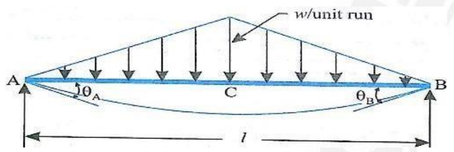
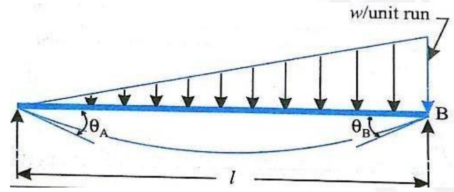
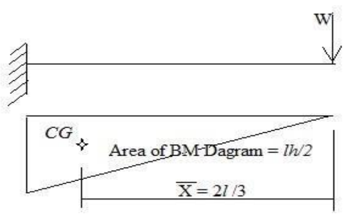
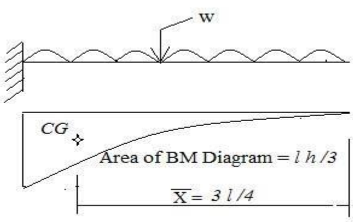
(SinceDissimply supported)

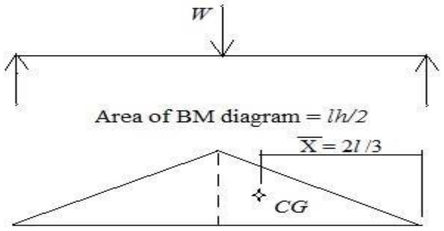
IMPORTANTTERMS

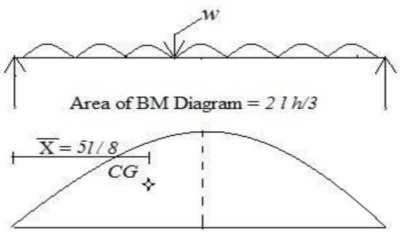
DESCRIPTION	SLOPE	DEFLECTION	MAX.BM
	$\theta_B = \frac{Wl^2}{2EI}$	$y_{BMax} = \frac{Wl^3}{3EI}$	$M_A = Wl$
	$\theta B = \theta C = \frac{Wa^2}{2\ EI}$	$y_C = \frac{Wa^3}{3\ E\ I}$ and $y_{BMax} = \frac{Wa^3}{3EI} + \frac{Wa^2}{3\ EI}(l-a)$	$M_A = Wa$

	$\theta_B = \frac{wl^3}{6EI}$	$y_{B\,Max} = \frac{wl^4}{8EI}$	$M_A = \frac{wl^2}{2}$
	$\theta B = \theta C = \frac{wa^3}{6\,EI}$	$y_C = \frac{wa^4}{8\,E\,I}$ and $y_{B\,Max} = \frac{wa^4}{8\,EI} + \frac{wa^3}{6\,EI}(l-a)$	$M_A = \frac{wa^2}{2}$
	$\theta_B = \frac{wl^3}{6\,EI} - \frac{w(l-a)^3}{6\,EI}$	$y_{B\,Max} = \frac{wl^4}{8EI} - \left[\frac{w(l-a)w(l-a)^3}{8\,E\,I} + \frac{}{6\,E\,I}a\right]$	$M_A = w(l-a)\left[\frac{l-a}{2} + a\right]$
	$\theta_B = \frac{wl^3}{24\,EI}$	$y_{B\,Max} = \frac{wl^4}{30\,EI}$	$M_A = \frac{wl^2}{6}$
	$\theta_B = \frac{Ml}{E\,I}$	$y_{B\,Max} = \frac{Ml^2}{2\,EI}$	$M_A = M$

	$\theta A = \theta B = \frac{Wl^2}{16\,EI}$	$y_{B\,Max} = \frac{Wl^3}{48\,EI}$	$M_C = \frac{Wl}{4}$

	$\theta_A = \frac{Wb(l^2-b^2)}{6EIl}$ $\theta_B = \frac{Wa(l^2-a^2)}{6EIl}$	$y_{max} = \frac{Wb(l^2-b^2)^{3/2}}{9\sqrt{3}EIl}$ *and* $y_C = \frac{Wa^2a^2}{3EIl}$	$M_C = \frac{Wab}{l}$
	$\theta A = \theta B = \frac{wl^3}{24EI}$	$y_{max} = \frac{5}{384}\frac{wl^4}{EI}$	$M_C = \frac{wl^2}{8}$
	$\theta_A = \frac{7}{360}\frac{wl^3}{EI}$ $\theta_B =$ 4	w 5 $y_{max} = \frac{2.5}{384}\frac{wl^4}{EI}$	$M_C = \frac{wl^3}{9\sqrt{3}}$
	$\theta_A = \theta_B = \frac{5}{192}\frac{Wl^2}{EI}$	$y_{max} = \frac{5}{120}\frac{wl^4}{EI}$	$M_C = \frac{wl^2}{12}$
MomentArea Method			
Slope	$\boldsymbol{\theta}$**= A/EI**	**Y=$A\overline{X}$/EI**	**Deflection**
A=AreaofBMdiagram;$\overline{X}$=Centreofgravity distance; **E=Young'smodulusofbeammaterial;I=MomentofInertiaofbeamsection**			
			





MECAULAY’SMETHOD

Step1: set the point XX at a distance of x m from free end/right support (OR) Near to fixed end/left support.

Step2:TakemomentaboutXX

Step3: integrate themoment equationwrtx with adding of constant C_1 at the first part of equation is slope equation

Step4: integrate again wrt x with adding another constant C_2 at the first part of equation is deflection equation

Step5:putconditionx=0;y=0;infirstpartofdeflection eqntofindC_2value(C_2=0)andPutanotherconditionx= l ; y = 0 in whole part of same eqn to find C_1 value.

Step6: then substitute C_1, C_2 value in slope & deflection Eqn to get real Slope and deflection eqn.

Step7: Now substitute required point distance in slope and deflection eqn in first part to find Near right support slope and deflection.

Step8: similarly include second part of eqn and substitute anotherxvaluetofindslopeand deflectionofeverypoint.

- $MI\,\frac{d^2y}{dx^2}$= Moment equation
- $MI\,\frac{dy}{dx} = Slope\ equation$
- MIy= $deflectionequation$
- Tofind maximum deflection put $\frac{dy}{dx} = 0$ to get x value and substitute x value in deflection to find max. deflection
- To evaluate integration in whole part
- In UDL, unload distancealsoassumeboth side UDL load act.

DOUBLEINTEGRATIONMETHOD

Sameprocedurefollowedasmecaulay’smethod But

some difference are

1. ConstantC_1,C_2areaddattheendof equation
2. Equationarenotseparatedonlywholeeqn used
3. Twoconditionareappliedinwholeeqntofind C_1, C_2
4. Ordnaryintegrationfollowed

THEORETICALQUESTIONS

1. Derive an expression for the slope and deflection of a beam subjected touniform bending moment.

2. Prove that the relation that $M = EI\ \frac{d^2y}{dx^2}$
where M=Bendingmoment,E=Young'smodulus,I=M.O.I.

3. Findanexpressionfortheslopeatthesupportsofasimplysupportedbeam, carrying a point load at the centre.

4. Provethatthedeflectionatthecentreofasimplysupportedbeam,carrying apointloadatthecentre,isgivenby$y_C = \frac{WL^3}{48EI}$whereW=Pointload,L=Length of beam.

5. Findanexpressionfortheslopeanddeflectionofasimplysupportedbeam, carrying a point load W at a distance 'a' from left support and at a distance 'b' from right support where a > b.

6. Prove that the slope and deflection of a simply supported beam of length L andcarryingauniformlydistributedloadofwperunitlengthovertheentirelength are given by
Slopeatsupports$= -\frac{WL^2}{24\,EI}$,and Deflectionatcentre$= \frac{5}{384}\frac{WL^3}{EI}$ WhereW =Total load =w×L.

7. What is Macaulay's method ? Where is it used ? Find an expression for deflectionatanysectionofasimplysupportedbeamwithaneccentricpointload, using Macaulay's method.

8. What is moment- area method ? Where is it convenientlyused ? Find the slope and deflection of a simply supported beam carrying a (i) point load at the centreand(ii)uniformlydistributedloadovertheentirelengthusingmoment-area method.

9. Whatisacantilever?Whatarethedifferentmethodsoffindingofslopeand deflection of a cantilever ?

10. Deriveanexpressionfortheslopeanddeflectionofacantileveroflength L, carrying a point load W at the free end by double integration method.

11. Solvequestions2,bymomentarea method.

12. Provethattheslopeanddeflectionofacantilevercarryinguniformlydistributed load over the whole length are given by,

$$\theta_B = \frac{wL^3}{6EI} \text{and} \quad y_B = \frac{wL^4}{8EI}$$

Wherew=UniformlydistributedloadandEI = Flexural rigidity.

13. Find the expression for the slope and deflection of a cantilever of length L which carries a uniformly distributed load over alength ‘a’ from the fixed end by (i)Doubleintegrationmethodand(ii)Momentareamethod.

14. Provethat theslope and deflection ofa cantileverlength L, which carries a gradually varying load from zero at the free end to w/m run at the fixed end are given by :

$$\theta_B = \frac{wL^3}{24EI} \text{and} \quad y_B = \frac{wL^4}{30EI}$$

WhereEI=Flexuralrigidity.

NUMERICALPROBLEMS

1. A wooden beam 4 m long, simply supported at its ends, is carrying a point load of 7.25 kN at its centre. The cross-section of the beam is 140 mm wide and 240mmdeep.IfEforthebeam=6×10^3N/mm^2,findthedeflectionatthecentre.

2. Abeam5mlong,simplysupportedatitsends,carriesapointloadWatits centre. If the slope at the ends of the beam is not to exceed 1^o, find the deflection at the centre of the beam.

3. Determine:(i)slopeattheleftsupport,(ii)deflectionundertheloadand (iii) maximum deflection of a simply supported beam of length 10 m, which is carrying a point load of 10 kN at a distance 6 m from the left end. TakeE=2×10^5N/mm^{2}andI=1×10^8mm^4.

4. A beam of uniform rectangular section 100 mm wide and 240 mm deep is simplysupportedatitsends.Itcarriesauniformlydistributedloadof9.125kN/m

run over the entire span of 4 m. Find the deflection at the centre if E = 1.1 × 10^4N/mm^2.

5. A beam of length 4.8 m and of uniform rectangular section is simply supportedatitsends.Itcarriesauniformlydistributedloadof9.375kN/mrunover theentirelength.Calculatethewidthanddepthofthebeamifpermissiblebending stress is 7 N/mm^2 and maximum deflection is not to exceed 0.95 cm.

TakeEforbeammaterial=1.05×10^4N/mm^2.

6. Solveproblem3,usingMacaulay'smethod.

7. Abeamoflength10missimplysupportedatitsendsandcarriestwopoint loadsof100kNand60kNatadistanceof2mand5mrespectivelyfromtheleft support. Calculate the deflections under each load. Find also the maximum deflection.

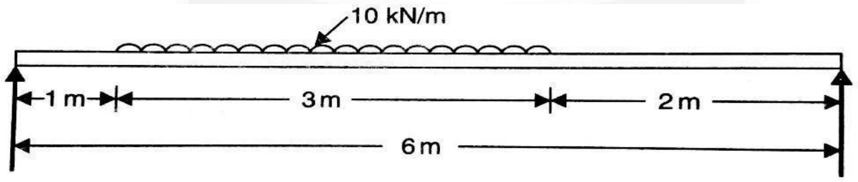
TakeI=18 ×10^8mm^{4}andE =2 ×10^5N/mm^2.

8. Abeamoflength20missimplysupportedatitsendsandcarriestwopoint loads of4 kN and10 kN at adistanceof8mand 12 mfromleftend respectively. Calculate : (i) deflection under each load (ii) maximum deflection.

TakeE=2 ×10^6N/mm^{2}andI=1×10^9mm^4.

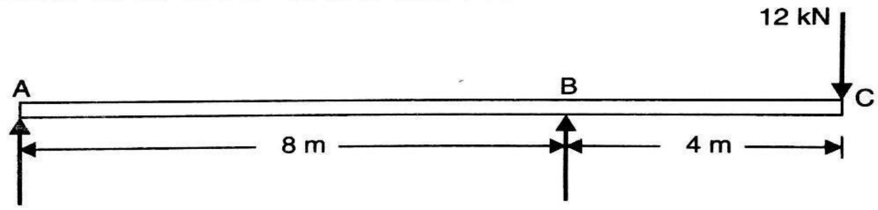
9. A beamof length6 missimply supported atits ends.It carries auniformly distributedloadof10kN/masshowninFig.Determinethedeflectionofthebeam at its mid-point and also the position and the maximum deflection.

TakeEI=4.5×10^8N/mm^2.



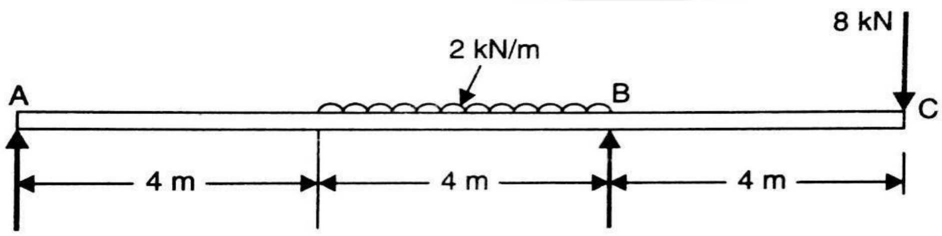
10. A beam ABC of length 12 metre has one support at the left end and other support atadistanceof8mfromtheleft end.Thebeamcarries apoint loadof12 kN at the right end as shown in Fig. Find theslopes overeach supports and at the right end. Find also the deflection at the right end.

TakeE=2 ×10^5N/mm^{2}andI=5×10^8mm^4.



11. An overhanging beam ABC is loaded as shown in Fig.. Determine the deflection of the beam at point C.

TakeE=2 $\times 10^5$N/mm^{2}andI=5$\times 10^8$mm^4.



12. A beam of span 8 m and of uniform flexural rigidity EI = 40 MN-m^2, is simplysupportedatitsends.Itcarriesauniformlydistributedloadof15kN/mrun over the entire span. It is also subjected to a clockwise moment of 160 kNm at a distance of 3 m from the left support. Calculate the slope of the beam at the point of application of the moment.

13. A cantilever of length 2 m carries a point load of 30 kN at the free end. If moment of inertia$I$ = 10^8 mm^4 and value of E = 2 $\times$ 10^5 N/mm^2, then find :

i) slopeofthecantileveratthefreeendand

ii) deflectionatthefreeend.

14. Acantileveroflength3mcarriesapointloadof60kNatadistanceof2m from the fixed end. If E = 2 $\times$ 10^5 N/mm^{2}and I = 10^8 mm^4, find :

i) slopeatthefreeendandii)deflection at the free end.

15. Acantileveroflength30mcarriesauniformlydistributedloadof24kN/m lengthovertheentirelength.Ifmomentofinertiaofthebeam=10^8mm^{3}andvalue of E = 2 $\times$ 10^5 N/mm^2, determine the slope and deflection at the free end.

16. Acantileveroflength3mcarriesauniformlydistributedloadovertheentire length.Iftheslopeatthefreeendis0.01777radians,findthedeflectionatthefree end.

17. Determine the slope and deflection at the free end of a cantilever of length 4 m which is carrying a uniformly distributed load of 12 kN/m over a length of 3 m fromthefixedend.Take EI=2×10^{13}N/mm^2.

18. A cantilever of length 3 m carries a uniformly distributed load of 15 kN/m over a length of 2 m from the free end. IfI = 10^8 $mm^4$and E = 2 × 10^5 N/mm^2, find :

i) slope at the free end andii) deflection at the free end.

19. Acantileveroflength2mcarriesaloadof20kNatthefreeendand30kN at adistance1 m from the end. Find the slope and deflection at thefreeend. Take E =2.0×10^5N/mm^2 and$I$=1.5 ×$10^8$$mm^4$.

20. Determine the deflection at the free end of a cantilever which is 2 m long andcarriesapointloadof9kNatthefreeendandauniformlydistributedloadof 8 kN/m over a length of 1 m from the fixed end.

21. Acantileveroflength2mcarriesauniformlyvaryingloadofzerointensity at the free end, and 45 kN/m at the fixed end. If E = 2 × 10^5 N/mm^2 and I = 10^8 mm^4, find the slope and deflection of the free end.

22. Acantileveroflength2mcarriesapointloadof30kNatthefreeend andanotherloadof30kNatitscentre.IfEI =10^{13}N/mm^2forthecantilever then determine by moment area method, the slope and deflection at the free endofcantilever.23.Acantileveroflength‘L’carriesaU.D.L.ofwperunit for a length of^{L}from 2 the fixed end. Determine the slope and deflection at the free end using area moment method.