

VECTOR SPACES

Definition :

Let F be a given field and let V be a non-empty set with addition and scalar multiplication rules applicable to any $u, v \in V$ such as a sum $u + v \in V$ and to any $u \in V, \alpha \in F$ a product $\alpha u \in V$. Then V is called a vector space over F if the following conditions hold :

1. Closure : for all $u, v \in V \Rightarrow u + v \in V$
2. Associative : $u + (v + w) = (u + v) + w \forall u, v, w \in V$.
3. Identity : $u + 0 = 0 + u = u$ for all $u \in V$, there exist $0 \in V$
4. Inverse : $(-u) + u = 0 = u + (-u)$ there exist $-u \in V$, for all $u \in V$
5. Commutative : $u + v = v + u$ for all $u, v \in V$
6. For all $\alpha \in F$ and for all $u \in V, \alpha u \in V$.
7. $\alpha(u + v) = \alpha u + \alpha v$, for all $\alpha \in F$ for all $u, v \in V$
8. $(\alpha + \beta)v = \alpha v + \beta v$, for all $\alpha, \beta \in F$ and for all $v \in V$
9. $(\alpha\beta)v = \alpha(\beta v)$, for all $\alpha, \beta \in F$ and for all $u, v \in V$
10. $1 \cdot v = v$ for all $v \in V$

Properties of vector space :

- (i) $\alpha \cdot 0 = 0, 0 \in V$, for all $\alpha \in F$
- (ii) $0 \cdot v = 0$, for all $v \in V, 0 \in F$

$$(iii) \quad (-\alpha)v = -(\alpha v) = \alpha(-v) \text{ for all } v \in F, v \in V$$

$$(iv) \quad \alpha v = 0, v \neq 0, \alpha = 0 \text{ where } \alpha \in F, \alpha \in V$$

$$(v) \quad \alpha(u - v) = \alpha v - \alpha v \text{ for all } \alpha \in F \text{ and } u, v \in V$$

Proof :

$$(i) \quad \text{since } 0+0=0 \text{ where } 0 \in V$$

$$\alpha(0 + 0) = \alpha 0 \text{ for all } \alpha \in F$$

$$\Rightarrow \alpha 0 + \alpha 0 = \alpha 0$$

$$\Rightarrow \alpha 0 + \alpha 0 = \alpha 0 + 0$$

$$\text{Hence } \alpha 0 = 0 \text{ [by left cancellation law]}$$

$$(ii) \quad \text{since } 0 + 0 = 0 \text{ where } 0 \in F$$

$$(0 + 0)v = 0v \text{ for all } v \in V$$

$$\Rightarrow 0v + 0v = 0v$$

$$\Rightarrow 0v + 0v = 0v + 0$$

$$\text{Hence } 0v = 0 \text{ [by left cancellation law]}$$

$$(iii) \quad (-\alpha)v = -(\alpha v) = \alpha(-v) \text{ for all } v \in F, v \in V$$

$$\text{Since } \alpha \in F \Rightarrow -\alpha \in F \text{ and } v \in V, -v \in V$$

$$\Rightarrow \alpha + (-\alpha) = 0 \in F ; v + (-v) = 0 \in V$$

$$\Rightarrow \alpha v + (-\alpha)v = [\alpha + (-\alpha)] v$$

$$\text{For all } v \in V ; \alpha v + \alpha(-v) = \alpha[v + (-v)]$$

$$\text{For all } \alpha \in F$$

$$\Rightarrow \alpha v + (-\alpha)v = 0v \text{ for all } v \in V ; \alpha v + \alpha(-v) = \alpha 0 \text{ for all } \alpha \in F$$

$\Rightarrow \alpha v + (-\alpha)v = 0$ for all $v \in V$; $\alpha v + \alpha(-v) = 0$ for all $\alpha \in F$

$\Rightarrow (-\alpha)v$ is the additive inverse of αv in V ; $\alpha(-v)$ is the additive inverse of αv in V .

$$(-\alpha)v = -(\alpha v) \quad ; \quad \alpha(-v) = -(\alpha v)$$

(iv) $\alpha v = 0, v \neq 0$

To prove $\alpha = 0$ where $\alpha \in F, v \in V$

Let $\alpha \neq 0$ then $\alpha^{-1} \in F$

Consider $\alpha v = 0$

$$\therefore \alpha^{-1}(\alpha v) = \alpha^{-1}(0)$$

$$\Rightarrow (\alpha^{-1}\alpha)v = 0$$

$$\Rightarrow 1v = 0$$

$$\Rightarrow v = 0 \text{ which is a contradiction.}$$

Hence $\alpha = 0$

Note : The vector space of V over the field F is denoted as $V(F)$.

- (i) C is a vector space over a field C and \mathbb{R}
- (ii) R is a vector space over a field \mathbb{R} but not in a field C
- (iii) Q is a vector space over a field Q .
- (iv) Z is not a vector space over a field R .
- (v) The set $\mathbb{R}^n = \{(a_1, a_2, \dots, a_n) \mid a_i \in \mathbb{R}\}$ is a vector space over \mathbb{R} .

(vi) The set $M_2(\mathbb{R})$ and $M_2(\mathbb{Q})$ of 2×2 matrices with entries from \mathbb{R} and \mathbb{Q} is a vector space over \mathbb{R} .

(vii) The set $Z_p[\mathbb{R}]$ of polynomials with coefficients from Z_p is a vector space over Z_p , where P is a prime.

(viii) Let E be a field and F be a subfield of E . Then E is a vector space over F .

(ix) Let $P_n(t)$ be the set of all polynomials $P(t)$ over a field F , where the degree of $P(t)$ is less than or equal to n . i.e.,

$$P(t) = a_0 + a_1 t + \dots + a_n t^n.$$

PROBLEMS UNDER VECTOR SPACE

Example 1. Prove that $R \times R$ is a vector space over R under addition and multiplication defined by $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and

$$a(x_1, x_2) = (ax_1, ax_2)$$

Sol: Let $x, y \in V = R \times R$

Then $x = (x_1, x_2)$

$$y = (y_1, y_2)$$

Where $x_1, x_2, y_1, y_2 \in R$

$$x + y = (x_1, x_2) + (y_1, y_2)$$

$$= (x_1 + y_1, x_2 + y_2) \in R \times R$$

Let $\alpha \in F$ and $x \in \mathcal{Y}$

$$\begin{aligned}\alpha x &= \alpha(x_1, x_2) \\ &= (\alpha x_1, \alpha x_2) \in R \times R.\end{aligned}$$

Therefore vector addition and scalar multiplications are true in $R \times R$.

1 Under addition

A_1 : Commutativity: $x + y = y + x, \forall x, y \in R \times R$

$$\begin{aligned}x + y &= (x_1, x_2) + (y_1, y_2) \\ &= (x_1 + y_1, x_2 + y_2) \\ &= (y_1 + x_1, y_2 + x_2) \\ &= (y_1, y_2) + (x_1, x_2) \\ &= y + x\end{aligned}$$

$$\therefore x + y = y + x, \forall x, y \in R \times R$$

A_2 : Associativity: $x + (y + z) = (x + y) + z, \forall x, y, z \in R \times R$

Let $x, y, z \in R \times R$. Then

$$x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2)$$

Where $x_1, x_2, y_1, y_2, z_1, z_2 \in R$

$$x + (y + z) = (x_1, x_2) + [(y_1, y_2) + (z_1, z_2)]$$

$$= (x_1, x_2) + (y_1 + z_1, y_2 + z_2)$$

$$= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2))$$

$$= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2)$$

$$= (x_1 + y_1, x_2 + y_2) + (z_1, z_2)$$

$$= ((x_1, x_2) + (y_1, y_2)) + (z_1, z_2)$$

$$= (x + y) + z$$

$$x + (y + z) = (x + y) + z, \forall x, y, z \in R \times R$$

A_3 : Existence of Identity: There exists $0 \in R \times R$ such that

$$x + 0 = x, \forall x \in R \times R$$

Let $0 \in R$. Then $0 = (0, 0) \in R \times R$.

$$x + 0 = (x_1, x_2) + (0, 0)$$

$$= (x_1 + 0, x_2 + 0)$$

$$= (x_1, x_2)$$

$$= x$$

$0 = (0, 0)$ is the zero element of $R \times R$

A_4 : Existence of Inverse: For all x in $R \times R$, there exists $-x \in R \times R$, such that

$$x + (-x) = 0$$

Let $x \in R \times R$

$\therefore x = (x_1, x_2)$, where $x_1, x_2 \in R$

Which implies $-x_1, -x_2 \in R$

$$\Rightarrow -x = (-x_1, -x_2) \in R \times R$$

$$x + (-x) = (x_1, x_2) + (-x_1, -x_2)$$

$$= (x_1 - x_1, x_2 - x_2)$$

$$= (0, 0)$$

$$= 0$$

$$x + (-x) = 0$$

\Rightarrow Inverse of x is $-x$

ie, inverse of (x_1, x_2) is $(-x_1, -x_2)$

II Under scalar multiplication:

$$M_1: a(x + y) = ax + ay; \forall a \in R \text{ and } \forall x, y \in R \times R$$

$$a(x + y) = a(x_1 + y_1, x_2 + y_2)$$

$$= (a(x_1 + y_1), a(x_2 + y_2))$$

$$= (ax_1 + ay_1, ax_2 + ay_2)$$

$$= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2)$$

$$= \alpha(x_1, x_2) + \alpha(y_1, y_2)$$

$$= \alpha x + \alpha y$$

$$\therefore a(x + y) = \alpha x + \alpha y \forall a \in R \text{ and } \forall x, y \in R \times R$$

$$M_2: (\alpha + \beta)x = \alpha x + \beta x, \forall \alpha, \beta \in R, \forall x \in R \times R$$

$$(\alpha + \beta)x = (\alpha + \beta)(x_1, x_2)$$

$$= ((\alpha + \beta)x_1, (\alpha + \beta)x_2)$$

$$= (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2)$$

$$= (\alpha x_1, \alpha x_2) + (\beta x_1, \beta x_2)$$

$$= \alpha(x_1, x_2) + \beta(x_1, x_2)$$

$$= \alpha x + \beta x$$

$$(\alpha + \beta)x = \alpha x + \beta x, \forall \alpha, \beta \in R, \forall x \in R \times R$$

$$M_3: a(\beta x) = (\alpha\beta)(x), \forall \alpha, \beta \in R, \forall x \in R \times R$$

$$\alpha(\beta x) = \alpha(\beta(x_1, x_2))$$

$$= \alpha(\beta x_1, \beta x_2)$$

$$= (\alpha(\beta x_1), \alpha(\beta x_2))$$

$$= ((\alpha\beta)x_1, (\alpha\beta)x_2)$$

$$= (\alpha\beta)(x_1, x_2)$$

$$= (\alpha\beta)(x)$$

$$\therefore \alpha(\beta x) = (\alpha\beta)(x) \forall \alpha, \beta \in R, \forall x \in R \times R$$

$M_4: 1 \cdot x = x, \forall x \in R \times R$ and $1 \in R$

$$1 \cdot x = 1(x_1, x_2)$$

$$= (1 \cdot x_1, 1 \cdot x_2)$$

$$= (x_1, x_2) = x$$

$1 \cdot x = x, \forall x \in R \times R$ and $1 \in R$

Therefore $V = R \times R$ is a vector space over R .

Example 2. Prove that F^n is a vector space over a field F under addition and

multiplication defined by $(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 +$

$y_2, \dots, x_n + y_n)$ and $\alpha(x_1, x_2, \dots, x_n) = (\alpha x_1, \alpha x_2, \dots, \alpha x_n)$

Let $x, y \in V = F^n$

Then $x = (x_1, x_2, \dots, x_n)$

$$y = (y_1, y_2, \dots, y_n)$$

where $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in F$

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \in F^n$$

Let $\alpha \in F$ and $x \in F^n$

$$\alpha x = \alpha(x_1, x_2, \dots, x_n)$$

$$= (\alpha x_1, \alpha x_2, \dots, \alpha x_n) \in F^n$$

Therefore vector addition and scalar multiplications are true in F^n .

I. Under addition

A_1 : Commutativity: $x + y = y + x, \forall x, y \in F^n$

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

$$= (y_1 + x_1, y_2 + x_2, \dots, y_n + x_n)$$

$$= (y_1, y_2, \dots, y_n) + (x_1, x_2, \dots, x_n)$$

$$= y + x$$

$$x + y = y + x, \forall x, y \in F^n$$

A_2 : Associativity: $x + (y + z) = (x + y) + z, \forall x, y, z \in F^n$

Let $x, y, z \in F^n$. Then

$$x = (x_1, x_2, \dots, x_n), y = (y_1, y_2, \dots, y_n), z = (z_1, z_2, \dots, z_n)$$

Where $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n, z_1, z_2, \dots, z_n \in F$

$$x + (y + z) = (x_1, x_2, \dots, x_n) + [(y_1, y_2, \dots, y_n) + (z_1, z_2, \dots, z_n)]$$

$$= (x_1, x_2, \dots, x_n) + (y_1 + z_1, y_2 + z_2, \dots, y_n + z_n)$$

$$= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), \dots, x_n + (y_n + z_n))$$

$$= ((x_1 + y_1) + z_1, (x_2 + y_2) + z_2, \dots, (x_n + y_n) + z_n)$$

$$= (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) + (z_1, z_2, \dots, z_n)$$

$$= ((x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n)) + (z_1, z_2, \dots, z_n)$$

$$= (x + y) + z$$

$$\therefore x + (y + z) = (x + y) + z, \forall x, y, z \in F^n$$

A_3 : Existence of Identity: There exists $0 \in F^n$ such that

$$x + 0 = 0 + x = x, \forall x \in F^n$$

Let $0 \in F$. Then $0 = (0, 0, \dots, 0) \in F^n$

$$x + 0 = (x_1, x_2, \dots, x_n) + (0, 0, \dots, 0)$$

$$= (x_1 + 0, x_2 + 0, \dots, x_n + 0)$$

$$= (x_1, x_2, \dots, x_n)$$

$$= x$$

$0 = (0, 0, \dots, 0)$ is the zero element of F^n

A_4 : Existence of Inverse: For all x in F^n , there exists $-x$ in F^n such that

$$(-x) + x = 0$$

Let $x \in F^n$.

$\therefore x = (x_1, x_2, \dots, x_n)$; where $x_1, x_2, \dots, x_n \in F$

Which implies $-x_1, -x_2, \dots, -x_n \in F$

$$-x = (-x_1, -x_2, \dots, -x_n) \in F^n$$

$$x + (-x) = (x_1, x_2, \dots, x_n) + (-x_1, -x_2, \dots, -x_n)$$

$$= (x_1 - x_1, x_2 - x_2, \dots, x_n - x_n)$$

$$= (0, 0, \dots, 0)$$

$$= 0$$

\Rightarrow Inverse of x is $-x$

ie, inverse of (x_1, x_2, \dots, x_n) is $(-x_1, -x_2, \dots, -x_n)$

II Under scalar multiplication:

$$M_1 = \alpha(x + y) = \alpha x + \alpha y, \forall \alpha \in F \text{ and } \forall x, y \in F^n$$

$$\alpha(x + y) = \alpha(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)$$

$$= (\alpha(x_1 + y_1), \alpha(x_2 + y_2), \dots, \alpha(x_n + y_n))$$

$$= (\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2, \dots, \alpha x_n + \alpha y_n)$$

$$= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2), \dots, (\alpha x_n + \alpha y_n)$$

$$= \alpha(x_1, x_2) + \alpha(y_1, y_2), \dots, (\alpha x_n + \alpha y_n)$$

$$= \alpha x + \alpha y$$

$$\therefore \alpha(x + y) = \alpha x + \alpha y, \forall \alpha \in F \text{ and } \forall x, y \in F^n$$

$$\mathbf{M}_2: (\alpha + \beta)x = \alpha x + \beta x, \alpha, \beta \in F, \forall x \in F^n$$

$$(\alpha + \beta)x = (\alpha + \beta)(x_1, x_2, \dots, x_n)$$

$$= ((\alpha + \beta)x_1, (\alpha + \beta)x_2, \dots, (\alpha + \beta)x_n)$$

$$= (\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2, \dots, \alpha x_n + \beta x_n)$$

$$= (\alpha x_1, \alpha x_2, \dots, \alpha x_n) + (\beta x_1, \beta x_2, \dots, \beta x_n)$$

$$= \alpha(x_1, x_2, \dots, x_n) + \beta(x_1, x_2, \dots, x_n)$$

$$= \alpha x + \beta x$$

$$\therefore (\alpha + \beta)x = \alpha x + \beta x, \forall \alpha, \beta \in F, \forall x \in F^n$$

$$\mathbf{M}_3: \alpha(\beta x) = (\alpha\beta)(x), \alpha, \beta \in F, \forall x \in F^n$$

$$\alpha(\beta x) = \alpha(\beta(x_1, x_2, \dots, x_n))$$

$$= \alpha(\beta x_1, \beta x_2, \dots, \beta x_n)$$

$$= (\alpha(\beta x_1), \alpha(\beta x_2), \dots, \alpha(\beta x_n))$$

$$= ((\alpha\beta)x_1, (\alpha\beta)x_2, \dots, (\alpha\beta)x_n)$$

$$= (\alpha\beta)(x_1, x_2, \dots, x_n)$$

$$= (\alpha\beta)(x)$$

$$\therefore \alpha(\beta x) = (\alpha\beta)(x), \forall \alpha, \beta \in F, \forall x \in F^n$$

$$M_4: 1 \cdot x = x, \forall x \in F^n \text{ and } 1 \in F$$

$$\begin{aligned} 1 \cdot x &= 1 \cdot (x_1, x_2, \dots, x_n) \\ &= (1 \cdot x_1, 1 \cdot x_2, \dots, 1 \cdot x_n) \\ &= (x_1, x_2, \dots, x_n) = x \end{aligned}$$

$$\therefore 1 \cdot x = x, \forall x \in F^n \text{ and } 1 \in F$$

$\therefore F^n$ is a vector space over F .

Example 3. Prove that set of complex numbers is a vector space over field

R .

$$\text{Sol: } V = C = \{(x + iy) / x, y \in R\}$$

Let $x, y \in C$

$$\text{Then } x = x_1 + iy_1, y = x_2 + iy_2$$

Where $x_1, y_1, x_2, y_2 \in R$

Addition of vectors is defined by

$$\begin{aligned} x + y &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= x_1 + x_2 + i(y_1 + y_2) \in C \end{aligned}$$

Scalar multiplication is defined by

For $\alpha \in R$ and $x \in C$

$$\alpha x = a(x_1 + iy_1)$$

$$= \alpha x_1 + i\alpha x_2 \in C$$

Therefore vector addition and scalar multiplications are true in C .

1. Under Addition

A_1 : Commutativity: $x + y = y + x, \forall x, y \in C$

$$\begin{aligned} x + y &= (x_1 + ix_2) + i(y_1 + iy_2) \\ &= (x_2 + ix_1) + i(y_2 + iy_1) \\ &= (x_2 + iy_2) + (x_1 + iy_1) \\ &= y + x \end{aligned}$$

$\therefore x + y = y + x, \forall x, y \in C$

A_2 : Associativity: $x + (y + z) = (x + y) + z, \forall x, y, z \in C$

Let $x, y, z \in C$

$\therefore x = x_1 + iy_1, y = x_2 + iy_2, z = x_3 + iy_3$

$$x + (y + z) = (x_1 + iy_1) + [(x_2 + iy_2) + (x_3 + iy_3)]$$

$$= (x_1 + iy_1) + [(x_2 + x_3) + i(y_2 + y_3)]$$

$$= (x_1 + (x_2 + x_3)) + i(y_1 + (y_2 + y_3))$$

$$= ((x_1 + x_2) + x_3) + i((y_1 + y_2)y_3)$$

$$= [(x_1 + x_2) + i(y_1 + y_2)] + (x_3 + iy_3)$$

$$= [(x_1 + iy_1) + (x_3 + iy_2)] + (x_3 + iy_3)$$

$$= (x + y) + z$$

$$x + (y + z) = (x + y) + z, \forall x, y, z \in \mathcal{C}$$

A_3 : Existence of Identity: There exists $0 \in \mathcal{C}$ such that

$$x + 0 = x, \forall x \in \mathcal{C}$$

Let $0 \in \mathcal{R}$. Then $0 = 0 + i0 \in \mathcal{C}$

$$x + 0 = (x_1 + iy_1) + (0 + i0)$$

$$= x_1 + 0 + i(y_1 + 0)$$

$$= x_1 + iy_1$$

$$= x$$

$0 = 0 + i0$ is the zero element of \mathcal{C}

A_4 : Existence of Inverse: For all x in \mathcal{C} , there exists $-x$ in \mathcal{C} such that

$$(-x) + x = 0$$

Let $x \in \mathbb{C}$. Then

$$x = x_1 + iy_1, \text{ where } x_1, y_1 \in \mathbb{R}$$

Which implies $-x_1, y_1 \in \mathbb{R}$

$$\therefore -x = -x_1 + i(-y_1) \in \mathbb{C}$$

$$x + (-x) = (x_1 + iy_1) + (-x_1 + i(-y_1))$$

$$= x_1 - x_1 + i(y_1 - y_1)$$

$$= 0 + i0$$

$$= 0$$

\therefore Inverse of x is $-x$

i.e. inverse of $x_1 + iy_1$ is $-x_1 + i(-y_1)$

II. Under scalar multiplication

$$M_1: \alpha(x + y) = \alpha x + \alpha y, \forall \alpha \in \mathbb{R} \text{ and } \forall x, y \in \mathbb{C}$$

$$\alpha(x + y) = \alpha[(x_1 + x_2) + i(y_1 + y_2)]$$

$$= \alpha(x_1 + x_2) + i\alpha(y_1 + y_2)$$

$$= (\alpha x_1 + \alpha x_2) + i(\alpha y_1 + \alpha y_2)$$

$$= (\alpha x_1 + i\alpha y_1) + (\alpha x_2 + i\alpha y_2)$$

$$= \alpha(x_1 + iy_1) + \alpha(x_2 + iy_2)$$

$$= \alpha x + \alpha y$$

$$\therefore \alpha(x + y) = \alpha x + \alpha y, \forall \alpha \in R \text{ and } \forall x, y \in C$$

$$M_2: (\alpha + \beta)x = \alpha x + \beta x, \forall \alpha, \beta \in R, \forall x \in C$$

$$(\alpha + \beta)x = (\alpha + \beta)(x_1 + iy_1)$$

$$= (\alpha + \beta)x_1 + i(\alpha + \beta)y_1$$

$$= \alpha x_1 + \beta x_1 + i(\alpha y_1 + \beta y_1)$$

$$= (\alpha x_1 + i\alpha y_1) + (\beta x_1 + i\beta y_1)$$

$$= \alpha(x_1 + iy_1) + \beta(x_1 + iy_1)$$

$$= \alpha x + \beta x$$

$$\therefore (\alpha + \beta)x = \alpha x + \beta x, \forall \alpha, \beta \in R, \forall x \in C$$

$$M_3: \alpha(\beta x) = (\alpha\beta)x, \forall \alpha, \beta \in R, \forall x \in C$$

$$\alpha(\beta x) = \alpha(\beta(x_1 + iy_1))$$

$$= \alpha(\beta x_1 + i\beta y_1)$$

$$= \alpha(\beta x_1) + i\alpha(\beta y_1)$$

$$= (\alpha\beta)x_1 + i(\alpha\beta)y_1$$

$$= (\alpha\beta)(x_1 + iy_1)$$

$$= (\alpha\beta)x$$

$$\therefore \alpha(\beta x) = (\alpha\beta)x, \forall \alpha, \beta \in R, \forall x \in C$$

$$M_4: 1 \cdot x = x, \forall x \in C \text{ and } 1 \in R$$

$$1 \cdot x = (1 + i0)(x_1 + iy_1)$$

$$= x_1 + iy_1$$

$$= x$$

$$\therefore 1 \cdot x^n = x, \forall x \in C \text{ and } 1 \in R$$

$\therefore C$ is a vector space over R .

