5.1 KINEMATICS – RECTLINEAR AND CURVILINEAR MOTION

Kinematics is the study of motion without reference to the cause of motion, i.e. force.

• Rectilinear motion is motion along a straight line.

• Throughout our discussion, all objects will be treated as particles.

• The rotational motion of objects is not considered. Only their translational motion in two dimensions either rectilinear or curvilinear will be discussed

Rectilinear kinematics deals with the following variables

a) Position

b) Displacement

c) Distance travelled

d) Velocity

e) Acceleration

a) Position:

• To define position of a particle moving along a straight line, only one co-ordinate is sufficient as that line can be chosen as the X or Y-axis.

• Thus motion of a particle in a straight line is a one dimensional problem.

• An origin has to be chosen on that line and a direction on one side of that origin has to be taken as positive.

• Generally, for a horizontal line (the X-axis) the right side of origin is taken positive and left side negative. For a vertical line, above the origin is taken positive and below the origin negative. For inclined line, one can use a sign convention according to convenience.

• **Position** is a vector quantity as it has both magnitude and direction.

• In one-dimension, vector quantities are represented by scalars with positive or negative sign representing their direction.

• For example, a position of - 5 m on a horizontal line indicates a position 5 m to the left of the origin.

b) Displacement:

• It is defined as the change in position. Displacement is also a vector quantity.

Displacement = Final position - Initial position.

• Note that the position is defined at a particular instant of time, whereas displacement is defined in a finite, non-zero time interval.

• The displacement and position at a particular time will be same if the particle starts from origin, i.e, the initial position is zero.

• For motion along the horizontal, if final position is to the right of the initial position the displacement will be positive. For vertical motion, displacement is positive when final position is above the initial position and negative when final position is below the initial position.

c) Distance travelled :

• It is positive scalar quantity which represents the total length of the path covered by the particle.

• It can be obtained from displacement.

• The magnitude of displacement in any time interval is equal to the distance travelled only when the particle keeps travelling in the same direction throughout that time interval.

• If the particle changes direction, split the time interval into suitable smaller intervals so that in each smaller interval, the partical travels in a particular direction.

• The total distance travelled can then be obtained by adding magnitudes of displacements in all these intervals.

• To illustrate the difference between the three quantities - position, displacement and distance travelled, consider a particle starting from position x_1 at time t_1, travelling to the right to reach position x_2 at time t_2 and then travelling to the left to reach position x_3 at time t_3 as shown in Fig. 9.1.1.

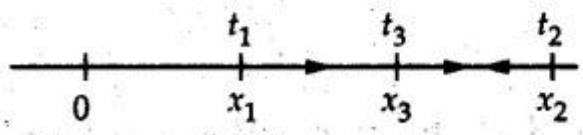


Fig. 9.1.1

• Then, at time t_3, position = x_3.

Displacement = x_3 - x_1

Distance travelled = $| x_2 - x_1 | + | x_3 - x_2 |$

d) Velocity:

• If a particle has displacement Ax in a time interval At, then the average velocity is given by

$$\boxed{v_{av} = \frac{\Delta x}{\Delta t}} \quad \ldots (9.1.1)$$

• The instantaneous velocity, which is generally referred to as velocity is given by instantaneous velocity =

$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$$\therefore \quad \boxed{v = \frac{dx}{dt}} \quad \ldots (9.1.2)$$

• Velocity is a vector quantity and in rectilinear motion will be represented by a scalar with positive or negative sign indicating its direction.

• The magnitude of velocity is known as speed.

e) Acceleration :

• If the velocity of particle changes by Δv in a time interval Δt, the average acceleration is given by

$$\boxed{a_{av} = \frac{\Delta v}{\Delta t}} \quad \ldots (9.1.3)$$

• The instantaneous acceleration, generally referred to as acceleration is given by

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

$$\therefore \quad \boxed{a = \frac{dv}{dt}} \quad \ldots (9.1.4)$$

• As $\quad v = \frac{dx}{dt}$

$$\boxed{a = \frac{d^2x}{dt^2}} \quad \ldots (9.1.5)$$

• The above equation can also be written as

$$a = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

$$\therefore \quad \boxed{a = v\frac{dv}{dx}} \quad \ldots (9.1.6)$$

• The acceleration is a vector quantity and in rectilinear motion, it will be represented by a scalar with positive or negative sign indicating its direction.

• The term **deceleration** is used to indicate that the speed (the magnitude of *v*) is decreasing.

• If '*v*' and '*a*' are in same direction it is acceleration whereas if '*v*' and 'a' are in opposite directions, there is deceleration.

• For example, when an object is thrown upwards (initial velocity upwards) then the object gets decelerated as the acceleration due to gravity is downwards. When an object is thrown downwards it gets accelerated as the initial velocity and the acceleration are both directed downwards.

• Thus if '*a*' and '*v*' have the same sign, it is acceleration and if they have opposite sign, it is deceleration.

• Problems in kinematics of rectilinear motion can be broadly classified into the following types:

i) Variable acceleration where functions are given relating any two of the four variables position, velocity, acceleration and time.

ii) Motion with constant acceleration. This type includes motion under gravity.

iii) Variable acceleration when functions are not known or the function changes from one-time interval to another. In such cases, motion diagrams are used which are graphs relating any two of the four variables.

iv) Dependent motion where variables for one object are related to variables of another object.

v) Relative motion.

2. Variable Acceleration

When functions relating any two of the four variables are given, use basic definitions (equations (9.1.2), (9.1.4), (9.1.5) and (9.1.6)) and differentiate or integrate the functions to obtain relations for the required variables.

• For example, if velocity is given as a function of time,

$v = f(t)$

then to get position 'x' in terms of 't', use $v = dx/dt = f(t)$,

• Separate the variables,

$dx = f(t)\, dt$

and integrate to obtain relation between 'x' and 't'.

• One can use either definite or indefinite integration. If definite integration is used, to obtain an equation for x in terms of t, use limits of integration as say x_0 to x for the variable 'x' and t_0 to t for the variable 't' where x_0 and t_0 are then two initial values.

$$\int_{x_0}^{x} dx = \int_{t_0}^{t} f(t)dt$$

• If indefinite integrals are used, the constant of integration has to be calculated usin some given conditions and then resubstituted to get the required equation.

• If acceleration is given as a function, either $a = dv/dt$ or

$a = dv/dx$ has to be used. In such cases we have to use that equation which enables us to separate the variables.

- **If $a = f_1(t)$ then $a = \frac{dv}{dt}$ has to be used. If $a = f_2(x)$ then $a = v\frac{dv}{dx}$ has to be used.**
- **If $a = f_3(v)$ then we have to use either $a = \frac{dv}{dt}$ or**

  $a = v\frac{dv}{dx}$ depending upon whether a relation between 'v' and 't' is required or a relation between 'v' and 'x'.

• While solving problems, the following concepts will be useful:

i) Displacement and position are same when starting position is origin. Hence in situations where starting position is not known, it is convenient to take the starting position as origin.

ii) The magnitude of displacement is equal to distance travelled only when particle keeps travelling in the same direction.

iii) Whenever particle changes its direction in rectilinear motion, its velocity at that instant becomes zero. At the point where particle changes direction, it reaches its maximum or minimum value of position co-ordinate 'x'.

 $dx/dt = 0$

 $v = 0$

iv) Condition for maximum or minimum velocity is

$dv / dt = 0$, i.e., $a = 0$

v) To calculate distance travelled, first put $v = 0$ to find whether particle changes direction or not in the given time interval.

vi) A linear relation between two variables, say when v varies linearly with t, is of the form

$v = mt + c$.

vii) When a vector quantity is proportional to another vector quantity, the constant of proportionality can be either positive or negative depending upon directions of the two vectors.

CURVILINEAR MOTION

The motion of a particle is said to be curvilinear when it moves along a curved path. Curvilinear motion can be either in two or three dimensions. We will be limiting our discussions only to the two dimensional curvilinear motion.

• This motion can be analyzed using rectangular components, normal and tangential components or radial and transverse components.

Position, Velocity and Acceleration in Curvilinear Motion

• For a particle moving along a curved path in a plane, the position, velocity and acceleration are vector quantities.

• Consider a particle moving along a curved path as shown in Fig. 9.5.1. If the particle is at point 'P' at time 't', then its position is defined by vector vector $\vec{r}$ which is directed from O to P.

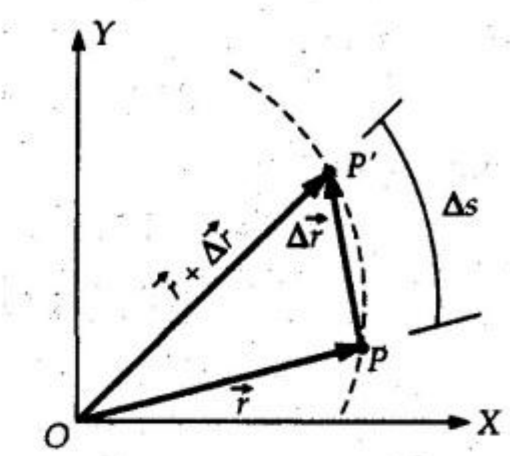


Fig. 9.5.1

• Let the particle be at position P' at time t + Δt.

The position vector at P' is $\vec{r} + \Delta\vec{r}$ where $\Delta\vec{r}$ is the change in position i.e. displacement of the particle represented by the vector $\vec{PP'}$.

• The distance travelled Δs is the length of arc PP'.

$$\boxed{\vec{v}_{av} = \frac{\Delta\vec{r}}{\Delta t}} \quad \text{... (9.5.1)}$$

will be the average velocity and the instantaneous velocity is

$$\boxed{\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}} \quad \text{... (9.5.2)}$$

• The magnitude of $\vec{v}$ is called speed.

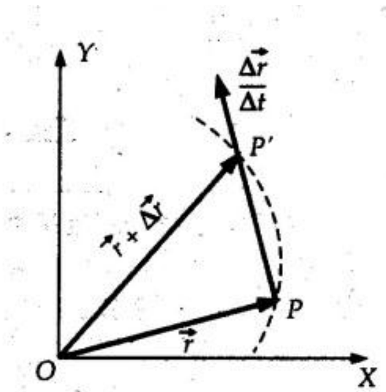


Fig. 9.5.2

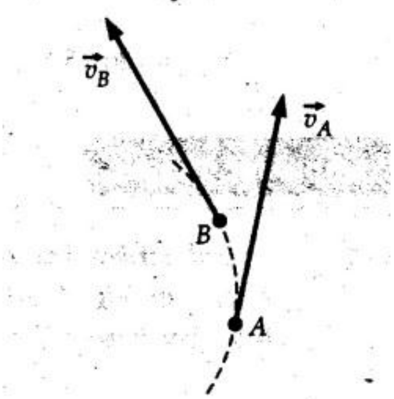


Fig. 9.5.3

As $\Delta t \rightarrow 0$, P' is very close to P and the direction of $\Delta \vec{r}$ is tangential to the curve.

• Hence velocity of the particle is always tangential.

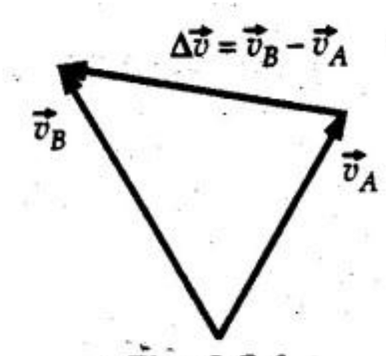


Fig. 9.5.4

• Consider velocity vectors at two points A and B as shown in Fig. 9.5.3. They are redrawn from a common point as shown in Fig. 9.5.4.

• The change in velocity is

$$\Delta \vec{v} = \vec{v}_B - \vec{v}_A$$

• The average acceleration is

$$\boxed{\vec{a}_{av} = \frac{\Delta \vec{v}}{\Delta t}} \quad \text{... (9.5.3)}$$

and instantaneous acceleration is

$$\boxed{\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}} \quad \text{... (9.5.4)}$$

• The three vectors are shown in Fig. 9.5.5. Note that the O velocity is tangential whereas acceleration is not.

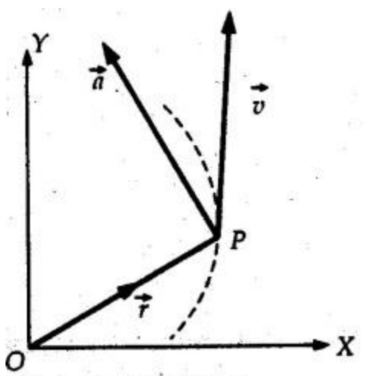


Fig. 9.5.5

Solved Examples

1.A ball is projected vertically upwards with a velocity of 20 m/s. Two seconds later, a second ball is projected vertically upwards with a velocity of 16 m/s. Find the height above the surface at which the two balls meet.

Solution:

For the first ball,

u = 20 m/s, a = -9.8 m/s^2

Let h = Height above the surface at which the two balls meet

t = Time for the first ball when they meet

 $s = h$

$$s = ut + \frac{1}{2}at^2 :$$

$$h = 20t + \frac{1}{2}(-9.81)t^2$$

$$\therefore \quad h = 20t - 4.905t^2 \qquad \text{... (1)}$$

For the second ball,

$$u = 16 \text{ m/s}, \quad s = h, \quad a = -9.81 \text{ m/s}^2,$$

$$\text{Time} = t - 2$$

$$s = ut + \frac{1}{2}at^2 :$$

$$h = 16(t-2) + \frac{1}{2}(-9.81)(t-2)^2$$

$$\therefore \quad h = 16t - 32 - 4.905(t^2 - 4t + 4)$$

$$\therefore \quad h = 35.62t - 4.905t^2 - 51.62 \qquad \text{... (2)}$$

From equations (1) and (2),

$20t - 4.905t^2 = 35.62t - 4.905t^2 - 51.62$

$51.62 = 15.62t$

 $t = 3.30474$

Substitute in equation (1)

$h = 20 \times 3.30474 - 4.905 \times 3.30474^2$

$$\therefore \quad \boxed{h = 12.526 \text{ m}}$$

2.A stone is dropped into a well. The sound of the splash is heard 3.63 seconds later. How far below the ground is the surface of water in the well? Assume the velocity of sound as 331 m/s.

Solution:

Let x = Depth of the well.

Then displacement of stone $s = -x$

Negative sign is for downward displacement.

$a = -g = -9.81$ m/s^2, t = Time and $u = 0$ for the stone.

$$s = ut + \frac{1}{2}at^2$$

$$\therefore \quad -x = 0 + \frac{1}{2}(-9.81)t^2$$

$$x = 4.905\, t^2 \quad \text{... (1)}$$

As total time is 3.63 s for stone to reach the water surface and sound to come up, the time for sound will be 3.63 - t.

As sound travels with constant velocity,

Distance travelled = Velocity × Time

$x = 331 \times (3.63 - t)$.

$$\therefore \quad x = 1201.53 - 331\, t \quad \text{... (2)}$$

From equations (1) and (2),

$4.905\, t^2 = 1201.53 - 331\, t$

$4.905t^2 + 331t - 1201.53 = 0$

 t = 3.4533 s, - 70.94 s

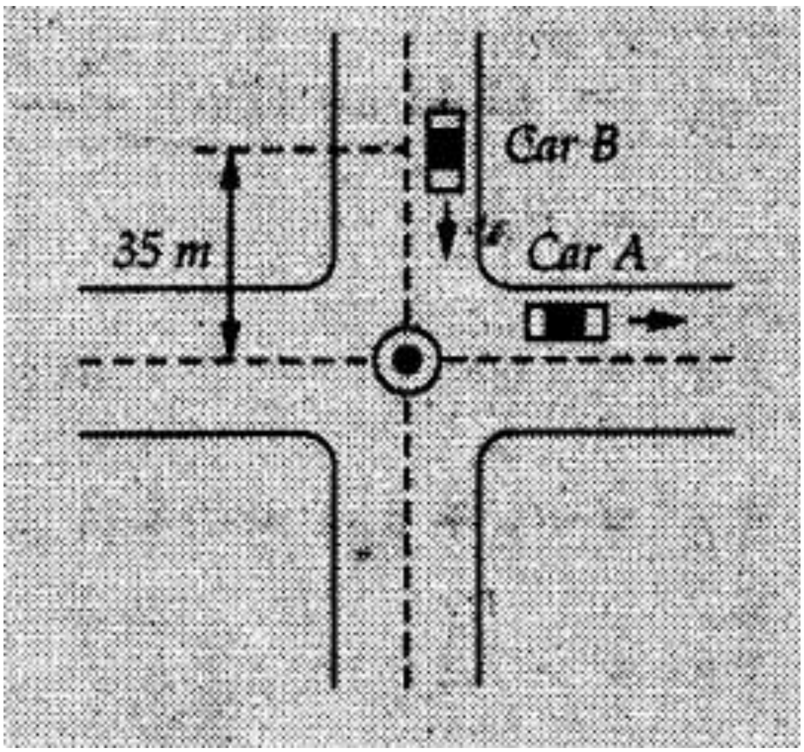
As t cannot be negative,

t = 3.4533 s

 From equation (1),

$$\boxed{x = 58.5 \text{ m}}$$

3.Two roads (N-S and E-W) cross at right angles at an intersection as shown in Fig.. Car A is travelling east at constant speed 36 km/h while car B, initially 35 m from intersection starts from rest and travels south at an acceleration 1.2 m/s^2 (uniform). Find position, velocity and acceleration of car B relative to car A six seconds after car A just crosses the intersection.



Solution:

At t = 0, car A is at the intersection and B is 35 m behind the intersection. At t = 6 s,

$$s_A = u_A \times t = \left(36 \times \frac{5}{18}\right)(6)$$

$$\therefore \quad s_A = 60 \text{ m}$$

$$s_B = u_B \cdot t + \frac{1}{2} a_B \cdot t^2 = 0 + \frac{1}{2}(1.2)(6^2)$$

$$\therefore \quad s_B = 21.6 \text{ m}$$

- At $t = 6$ m, distance of B from the intersection will be 35 - 21.6 = 13.4 m

The position vectors of A and B at $t = 6$ can be written, taking the intersection as origin, as :

$$\vec{v}_A = 60\hat{i}$$

$$\vec{r}_B = 13.4\,\hat{j}$$

Position of B with respect to A is

$$\vec{r}_{B/A} = \vec{r}_B - \vec{r}_A$$

$$\vec{r}_{B/A} = 13.4\,\hat{j} - 60\,\hat{i}$$

$$\therefore \quad \vec{r}_{B/A} = -60\,\hat{i} + 13.4\,\hat{j}$$

$$r_{B/A} = \sqrt{60^2 + 13.4^2} = 61.48 \text{ m}$$

$$\theta = \tan^{-1}\left(\frac{13.4}{60}\right) = 12.6^\circ$$

- Position of B w.r.t. A is

$$\boxed{\vec{r}_{B/A} = 61.48 \text{ m}, 12.6^\circ}$$

At $t = 6$ s

$$\vec{v}_A = 36 \times \frac{5}{18}\hat{i} = 10\,\hat{i} \text{ m/s}$$

$$v_B = u_B + a_B t = 0 + (1.2)(6) = 7.2 \text{ m/s}$$

$$\therefore \quad \vec{v}_B = -7.2\hat{j} \text{ m/s}$$

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A = (-7.2\hat{j}) - (10\hat{i})$$

$$\therefore \quad \vec{u}_{B/A} = -10\hat{i} - 7.2\hat{j}$$

$$v_{B/A} = \sqrt{10^2 + 7.2^2} = 12.32 \text{ m/s}$$

$$\theta = \tan^{-1}\left(\frac{7.2}{10}\right) = 35.75^\circ$$

$$\therefore \quad \boxed{v_{B/A} = 12.32 \text{ m/s}, 35.75^\circ}$$

$$\vec{a}_A = 0; \quad \vec{a}_B = -1.2\hat{j}$$

$$\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A = -1.2\hat{j}$$

$$\therefore \quad \boxed{a_{B/A} = 1.2 \text{ m/s}^2 \downarrow}$$

4.Two stones A and B are projected from the same point at inclinations of 45° and 30° respectively to the horizontal. Find the ratio of the velocities of projection of A and B if the maximum height reached by them is the same.

Solution:

The maximum height reached by a projectile is given by

$$H = \frac{u^2 \sin^2 \alpha}{2g}$$

For A, $u = u_A$, $\alpha = 45°$

$$\therefore \quad H = \frac{u_A^2 \sin^2 45}{2g} \quad \text{... (1)}$$

For B, $u = u_B$, $\alpha = 30°$

$$\therefore \quad H = \frac{u_B^2 \sin^2 30}{2g} \quad \text{... (2)}$$

Dividing equation (1) by equation (2),

$$1 = \frac{u_A^2 \sin^2 45}{u_B^2 \sin^2 30}$$

$$\sin 30 = \frac{1}{2} \quad \text{and} \quad \sin 45 = \frac{1}{\sqrt{2}}$$

$$\therefore \quad 1 = \frac{u_A^2 \times \frac{1}{2}}{u_B^2 \times \frac{1}{4}}$$

$$\therefore \quad \frac{u_A^2}{u_B^2} = \frac{1}{2}$$

$$\therefore \quad \boxed{\frac{u_A}{u_B} = \frac{1}{\sqrt{2}}}$$

5. A particle is projected with an initial velocity of 60 m/s, at an angle of 75° with the horizontal. Determine

a) The maximum height attained by the particle

b) Horizontal range of particle

c) Time taken by the particle to reach highest point

d) Time of flight

Solution :

a) Maximum height $H = \frac{u^2 \sin^2 \alpha}{2g} = \frac{60^2 \sin^2 75}{2 \times 9.81}$

$\therefore$ $\boxed{H = 171.2 \text{ m}}$

b) Horizontal range $R = \frac{u^2 \sin 2\alpha}{g} = \frac{60^2 \sin(2 \times 75)}{9.81}$

$\therefore$ $\boxed{R = 183.49 \text{ m}}$

c) Time taken by the particle to reach highest point is $t = \frac{T}{2} = \frac{u \sin \alpha}{g}$

$\therefore$ $t = \frac{60 \sin 75}{9.81}$

$\therefore$ $\boxed{t = 5.9 \text{ s}}$

d) Time of flight $T = 2\,t = 2 \times 5.9$

$\therefore$ $\boxed{T = 11.8 \text{ s}}$

A cricket ball thrown from a height of 1.8 above ground level at an angle of 30° with the horizontal with a velocity of 12 m/s is caught by a fielder at a height of 0.6 m above the ground. Determine the distance between the two players.

Solution:

The equation of path is,

$$y = x \tan\alpha - \frac{g x^2}{2u^2 \cos^2 \alpha}$$

Here $y = -1.2, \quad x = ?$

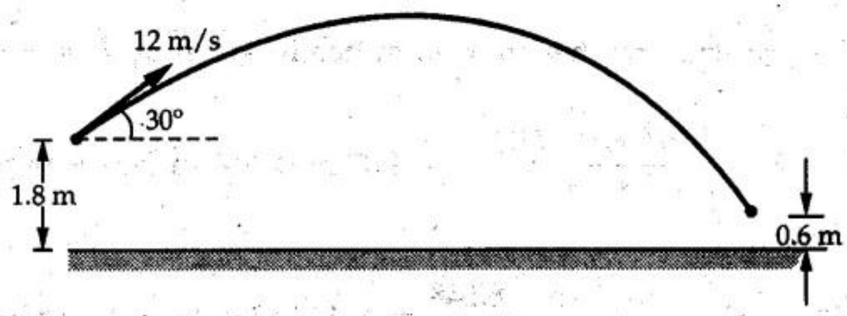


Fig. 9.7.3

$g = 9.81, \quad \alpha = 30°, \quad u = 12$ m/s

$$\therefore \quad -1.2 = x \tan 30 - \frac{9.81 x^2}{2 \times 12^2 \cos^2 30}$$

$$\therefore \quad \frac{9.81 x^2}{2 \times 144 \cos^2 30} - \tan 30 \times x - 1.2 = 0$$

$$\therefore \quad x = 14.53 \text{ m}, \; -1.82 \text{ m}$$

x cannot be negative

$$\boxed{x = 14.53 \text{ m}}$$