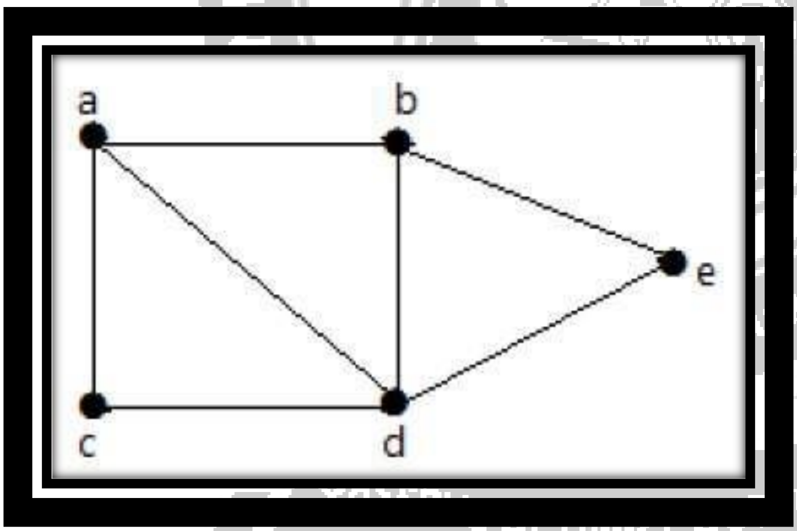
Connectivity:

A graph is said to be connected if there is a path between every pair of vertex. From every vertex to any other vertex, there should be some path to traverse. That is called the connectivity of a graph. A graph with multiple disconnected vertices and edges is said to be disconnected.

Example 1

In the following graph, it is possible to travel from one vertex to any other vertex. For example, one can traverse from vertex 'a' to vertex 'e' using the path 'a-b-e'.



Theorem: 1

Show that graph G is disconnected if and only if its vertex set V can be partitioned into two nonempty subsets V_1 and V_2 such that there exists no edge in G whose one end vertex is in V_1 and the other in V_2.

Proof:

Suppose that such a partitioning exists. Consider two arbitrary vertices a and b of G such that $a \in V_1$ and $b \in V_2$.

No path can exist between vertices a and b.

Otherwise, there would be atleast one edge whose one end vertex be in V_1 and the other in V_2.

Hence if partition exists, G is not connected.

Conversely, let G be a disconnected graph.

Consider a vertex a in G.

Let V_1 be the set of all vertices that are joined by paths to a.

Since G is disconnected, V_1 does not include all vertices of G.

The remaining vertices will form a set V_2.

No vertex in V_1 is joined to any in V_2 by an edge.

Hence the partition.

Hence the proof.

Components of a graph:

The connected subgraphs of a graph G are called components of the graph G.

Theorem: 1

A simple graph with n vertices and k components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

Proof:

Let $n_1, n_2, \dots, n_k$ be the number of vertices in each of k components of the graph G.

Then $n_1 + n_2 + \dots + n_k = n = |V(G)|$

$\sum_{i=1}^{k} n_i = n \quad \dots (1)$

Now, $\sum_{i=1}^{k}(n_i - 1) = (n_1 - 1) + (n_2 - 1) + \dots + (n_k - 1)$

$= \sum_{i=1}^{k} n_i - k$

$\Rightarrow \sum_{i=1}^{k}(n_i - 1) = n - k$

Squaring on both sides

$\Rightarrow \left[\sum_{i=1}^{k}(n_i - 1)\right]^2 = (n-k)^2$

$\Rightarrow (n_1 - 1)^2 + (n_2 - 1)^2 + \dots + (n_k - 1)^2 \leq n^2 + k^2 - 2nk$

$\Rightarrow {n_1}^2 + 1 - 2n_1 + {n_2}^2 + 1 - 2n_2 + \dots + {n_k}^2 + 1 - 2n_k \leq n^2 + k^2 - 2nk$

$\Rightarrow \sum_{i=1}^{k} {n_i}^2 + k - 2n \leq n^2 + k^2 - 2nk$

$\Rightarrow \sum_{i=1}^{k} {n_i}^2 \leq n^2 + k^2 - 2nk + 2n - k$

$\Rightarrow \sum_{i=1}^{k} {n_i}^2 = n^2 + k^2 - k - 2nk + 2n$

$= n^2 + k(k-1) - 2n(k-1)$

$= n^2 + (k-1)(k-2n) \quad ...(2)$

Since, G is simple, the maximum number of edges of G in its components is $\frac{n_i(n_i-1)}{2}$.

Maximum number of edges of $G = \sum_{i=1}^{k} \frac{n_i(n_i-1)}{2}$

$= \sum_{i=1}^{k} \left[\frac{{n_i}^2 - n_i}{2}\right]$

$= \frac{1}{2}\sum_{i=1}^{k} {n_i}^2 - \frac{1}{2}\sum_{i=1}^{k} n_i$

$\leq \frac{1}{2}[n^2 + (k-1)(k-2n)] - \frac{n}{2}$ (Using (1) and (2))

$= \frac{1}{2}[n^2 - 2nk + k^2 + 2n - k - n]$

$= \frac{1}{2}[n^2 - 2nk + k^2 + n - k]$

$= \frac{1}{2}[(n-k)^2 + (n-k)]$

$= \frac{1}{2}[(n-k)(n-k+1)]$

Maximum number of edges of $G \leq \frac{(n-k)(n-k+1)}{2}$

Hence the proof.