DESIGN OF FIR FILTERS USING WINDOWING TECHNIQUES

Design of FIR Filter:

FIR Filter can be designed using three following techniques.

1. Fourier series method
2. Windowing technique
3. Frequency sampling method.

Filter design using windowing technique:

Explain the designing of FIR filters using windows. [April/May-2011]

The desired frequency response of any digital filter is periodic in frequency and can be expanded in a Fourier series.

$$H_d(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h_d(n)e^{-j\omega n} \text{ ------------------------>(1)}$$

Where,

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\, d\omega \text{ ------------------------>(2)}$$

Gibb's Phenomenon:

One possible way of finding an FIR filter that approximates $H(e^{j\omega})$ would be truncate the infinite Fourier series at $n = \pm\left(\frac{M-1}{2}\right)$. Abrupt truncation of the series will lead to oscillation both pass band and stop band.

This phenomenon is known as Gibbs phenomenon.

Types of window:

- Rectangular window.
- Hanning window.
- Hamming window.

Rectangular window:

The rectangular window sequence is given by,

$$w_R(n) = \begin{cases} 1 & for -\left(\frac{M-1}{2}\right) \le n \le \left(\frac{M-1}{2}\right) \\ 0 & for\ otherwise. \end{cases}$$

Hanning window:

The hanning window sequence can be obtained by

$$w_{Hn}(n) = \begin{cases} 0.5 + 0.5\cos\frac{2\pi n}{M-1} & for -\left(\frac{M-1}{2}\right) \le n \le \left(\frac{M-1}{2}\right) \\ 0 & for\ otherwise. \end{cases}$$

Hamming window:

The hamming window can be obtained by

$$w_{Hm}(n) = \begin{cases} 0.54 + 0.46\cos\dfrac{2\pi n}{M-1} & for -\left(\dfrac{M-1}{2}\right) \le n \le \left(\dfrac{M-1}{2}\right) \\ 0 & for\ otherwise. \end{cases}$$

Filter coefficient ($h_d(n)$) for different types of Filters:

Type of Filter	$h_d(n)$
LPF	$h_d(n) = \dfrac{\omega_C}{\pi}\ for\ n = \alpha$ $h_d(n) = \dfrac{\sin \omega_C (n-\alpha)}{\pi(n-\alpha)}\ for\ n \ne \alpha$
HPF	$h_d(n) = 1 - \dfrac{\omega_C}{\pi}\ for\ n = \alpha$ $h_d(n) = \dfrac{1}{\pi(n-\alpha)}\left[\sin(n-\alpha)\pi - \sin(n-\alpha)\omega_C\right] for\ n \ne \alpha$
BPF	$h_d(n) = \dfrac{\omega_{C2} - \omega_{C1}}{\pi}\ for\ n = \alpha$ $h_d(n) = \dfrac{1}{\pi(n-\alpha)}\left[\sin \omega_{C2}(n-\alpha) - \sin \omega_{C1}(n-\alpha)\right] for\ n \ne \alpha$
BSF	$h_d(n) = 1 - \left[\dfrac{\omega_{C2} - \omega_{C1}}{\pi}\right] for\ n = \alpha$ $h_d(n) = \dfrac{1}{\pi(n-\alpha)}\left[\sin\omega_{C1}(n-\alpha) - \sin\omega_{C2}(n-\alpha) + \sin(n-\alpha)\pi\right] for\ n \ne \alpha$

Design an ideal low pass filter with a frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1\ for\ -\dfrac{\pi}{2} \le \omega \le \dfrac{\pi}{2} \\ 0\ \ for\ \ \dfrac{\pi}{2} \le |\omega| \le \pi \end{cases}$$

Find the values of *h(n)* for M=11 using hanning window. Find H(z). Plot the magnitude and frequency response. (May/June-14)(Nov/Dec-14) (April/May 2011)(April/May-08) (Nov/Dec-09) (Nov/Dec-10)

Step 1: To find filter coefficient.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.e^{j\omega n}\, d\omega$$

$$= \frac{\sin(n-\alpha)\omega_C}{\pi(n-\alpha)}; \quad \because \alpha = 0$$

$$h_d(n) = \frac{\sin\frac{\pi}{2}n}{\pi n}; \quad -5 \le n \le 5$$

$$For\ n=0; \quad h_d(0) = \frac{\sin\frac{\pi}{2}(0)}{\pi(0)}$$

$$= \frac{1}{2}\frac{\sin\frac{\pi n}{2}}{\frac{\pi n}{2}} \qquad\qquad \therefore \lim_{\theta->0}\frac{\sin\theta}{\theta}=1$$

$$h_d(0) = \frac{1}{2}$$

$$For\ n=1; \quad h_d(1)=h_d(-1)=\frac{\sin\frac{\pi}{2}(1)}{\pi(1)}=\frac{1}{\pi}=0.3183$$

$$For\ n=2; \quad h_d(2)=h_d(-2)=\frac{\sin\pi}{2\pi}=0$$

$$For\ n=3; \quad h_d(3)=h_d(-3)=\frac{\sin\frac{3\pi}{2}}{3\pi}=-0.106$$

$$For\ n=4; \quad h_d(4)=h_d(-4)=\frac{\sin 2\pi}{4\pi}=0$$

$$For\ n=5; \quad h_d(5)=h_d(-5)=\frac{\sin\frac{5\pi}{2}}{5\pi}=0.06366$$

Step 2: To find hanning window:

$$w_{Hn}(n)=\begin{cases}0.5+0.5\cos\frac{2\pi n}{M-1} & for -\left(\frac{M-1}{2}\right)\le n\le\left(\frac{M-1}{2}\right)\\ 0 \quad for\ otherwise.\end{cases}$$

$$=0.5+0.5\cos\frac{2\pi n}{10} \qquad for -5\le n\le(5)$$

$$For\ n=0; \quad w_{Hn}(0)=0.5+0.5\cos\frac{\pi(0)}{5}=1$$

$$For\ n=1; \quad w_{Hn}(1)=w_{Hn}(-1)=0.5+0.5\cos\frac{\pi(1)}{5}=0.9045$$

$$For\ n=2; \quad w_{Hn}(2)=w_{Hn}(-2)=0.5+0.5\cos\frac{\pi(2)}{5}=0.6545$$

$$For\ n=3; \quad w_{Hn}(3)=w_{Hn}(-3)=0.5+0.5\cos\frac{\pi(3)}{5}=0.3454$$

$$For\ n=4; \quad w_{Hn}(4)=w_{Hn}(-4)=0.5+0.5\cos\frac{\pi(4)}{5}=0.0954$$

$$For\ n=5; \quad w_{Hn}(5)=w_{Hn}(-5)=0.5+0.5\cos\frac{\pi(5)}{5}=0$$

Step 3: To find filter coefficients using hanning window are

$$h(n)=h_d(n)*w_{Hn}(n) \qquad for -5\le n\le 5$$

$$h(0)=h_d(0)*w_{Hn}(0)=(0.5)*1=0.5$$

$$h(1)=h_d(1)*w_{Hn}(1)=0.3183*0.9045=0.2879$$

$$h(2)=h_d(2)*w_{Hn}(2)=0*0.6545=0$$

$$h(3)=h_d(3)*w_{Hn}(3)=-0.106*0.3454=0.0366$$

$$h(4)=h_d(4)*w_{Hn}(4)=0.0636*0.0954=0.00606$$

$$h(5)=h_d(5)*w_{Hn}(5)=0*0.65=0$$

Step 4: The transfer function of the filter is given by

$$H(z) = h(0) + \sum_{n=1}^{\frac{M-1}{2}} \left[h(n)\left(z^{n} + z^{-n}\right)\right]$$

$$= 0.5 + \sum_{n=1}^{5} h(n)\left(z^{n} + z^{-n}\right)$$

$$= 0.5 + h(1)\left(z^{1} + z^{-1}\right) + h(2)\left(z^{2} + z^{-2}\right) + h(3)\left(z^{3} + z^{-3}\right) + h(4)\left(z^{4} + z^{-4}\right) + h(5)\left(z^{5} + z^{-5}\right)$$

$$= 0.5 + 0.287z^{1} + 0.287z^{-1} + 0.0366z^{3} + 0.0366z^{-3} + 0.006z^{4} + 0.006z^{-4}$$

Step 5 : The transfer function of the realizable filter is

$$H'(z) = z^{-\frac{(M-1)}{2}} H(z)$$

$$= z^{-5}\left[0.5 + 0.287z^{1} + 0.287z^{-1} + 0.0366z^{3} + 0.0366z^{-3} + 0.006z^{4} + 0.006z^{-4}\right]$$

$$H'(z) = 0.5z^{-5} + 0.287z^{-4} + 0.287z^{-6} + 0.0366z^{-2} + 0.0366z^{-8} + 0.006z^{-1} + 0.006z^{-9}$$

The filter coefficients of causal filter are given by

$$h(0) = h(3) = h(7) = h(10) = 0, h(1) = h(9) = 0.006; h(2) = h(8) = 0.0366; h(4) = h(6) = 0.287; h(5) = 0.5$$

Step 6 : The frequency response is given by

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{5} a(n)\cos\omega n$$

where,

$$a(0) = h\left(\frac{M-1}{2}\right) = h(5) = 0.5$$

$$a(n) = 2h\left(\frac{M-1}{2} - n\right)$$

$$a(1) = 2h(5-1) = 2h(4) = 2(0.006) = 0.12$$

$$a(2) = 2h(5-2) = 2h(3) = 0$$

$$a(3) = 2h(5-3) = 2h(2) = 2*0.036 = 0.072$$

$$a(4) = 2h(5-4) = 2h(1) = 2*0.006 = 0.12$$

$$a(5) = 2h(5-5) = 2h(0) = 0$$

$$\bar{H}(e^{j\omega}) = \sum_{n=0}^{5} a(n)\cos\omega n$$

$$= a(0) + a(1)\cos\omega + a(2)\cos 2\omega + a(3)\cos 3\omega + a(4)\cos 4\omega + a(5)\cos 5\omega$$

$$= 0.5 + 0.12\cos\omega + 0.072\cos 3\omega + 0.12\cos 4\omega$$

Magnitude in dB is calculated by varying 0 to 10 and tabulated below.

ω(in degree)	0	1	2	3	4	5
$\bar{H}(e^{j\omega})$	0.812	0.8115	0.810	0.8083	0.8054	0.8018
$\lvert H(e^{j\omega})\rvert_{dB}$	-1.8	-1.814	-1.83	-1.85	-1.88	-1.91

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Design an ideal high pass filter with a frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1 & for\ \frac{\pi}{4} \le |\omega| \le \pi \\ 0 & for\ |\omega| \le \frac{\pi}{4} \end{cases}$$

Find the values of *h(n)* for N=11 using hanning window. (May/June-16)(April/May-08)

Solution:

Given :

$$H_d(e^{j\omega}) = \begin{cases} 1 & for\ \frac{\pi}{4} \le |\omega| \le \pi \\ 0 & for\ |\omega| \le \frac{\pi}{4} \end{cases} \qquad Hence\ \omega_C = \frac{\pi}{4}$$

Step 1: To find filter coefficient.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.e^{j\omega n}\, d\omega$$

$$h_d(n) = \frac{1}{\pi(n-\alpha)}\left[\sin(n-\alpha)\pi - \sin(n-\alpha)\omega_C\right] for\ n \ne \alpha$$

$$h_d(n) = \frac{1}{\pi n}\left[\sin n\pi - \sin\frac{n\pi}{4}\right];\quad -5 \le n \le 5$$

Step 2 :

$$For\ n=0 \qquad h_d(0) = \frac{1}{\pi(0)}\left[\sin(0)\pi - \sin\frac{(0)\pi}{4}\right]$$

$$U\sin g,\ L'hospital\ rule \qquad h_d(0) = 1 - \frac{1}{4} = 0.75$$

$$For\ n=1; \qquad h_d(1) = h_d(-1) = \frac{1}{\pi(1)}\left[\sin(1)\pi - \sin\frac{(1)\pi}{4}\right] = -0.225$$

$$For\ n=2; \qquad h_d(2) = h_d(-2) = \frac{1}{\pi(2)}\left[\sin(2)\pi - \sin\frac{(2)\pi}{4}\right] = -0.159$$

$$For\ n=3; \qquad h_d(3) = h_d(-3) = \frac{1}{\pi(3)}\left[\sin(3)\pi - \sin\frac{(3)\pi}{4}\right] = -0.075$$

$$For\ n=4; \qquad h_d(4) = h_d(-4) = \frac{1}{\pi(4)}\left[\sin(4)\pi - \sin\frac{(4)\pi}{4}\right] = 0$$

$$For\ n=5; \qquad h_d(5) = h_d(-5) = \frac{1}{\pi(5)}\left[\sin(5)\pi - \sin\frac{(5)\pi}{4}\right] = 0.045$$

Step 2: Using Hanning window:

$$w_{Hn}(n) = \begin{cases} 0.5 + 0.5\cos\frac{2\pi n}{M-1}, & \left(\frac{M-1}{2}\right) \le n \le \left(\frac{M-1}{2}\right) \\ 0 & otherwise \end{cases}$$

$$w_{Hn}(0) = 0.5 + 0.5 = 1$$

$$w_{Hn}(1) = w_{Hn}(-1) = 0.5 + 0.5\cos\left(\frac{\pi(1)}{5}\right) = 0.9045$$

$$w_{Hn}(2) = w_{Hn}(-2) = 0.5 + 0.5\cos\left(\frac{\pi(2)}{5}\right) = 0.655$$

$$w_{Hn}(3) = w_{Hn}(-3) = 0.5 + 0.5\cos\left(\frac{\pi(3)}{5}\right) = 0.345$$

$$w_{Hn}(4) = w_{Hn}(-4) = 0.5 + 0.5\cos\left(\frac{\pi(4)}{5}\right) = 0.0945$$

$$w_{Hn}(5) = w_{Hn}(-5) = 0.5 + 0.5\cos\left(\frac{\pi(5)}{5}\right) = 0$$

Step 3: The filter coefficients using hanning window are,

$$h(n) = h_d(n)w_{Hn}(n) \quad for -5 \le n \le 5$$

$$h(0) = h_d(0)w_{Hn}(0) = (0.75)(0) = 0$$
$$h(1) = h_d(1)w_{Hn}(1) = (-0.225)(0.905) = -0.204$$
$$h(2) = h_d(2)w_{Hn}(2) = (-0.159)(0.655) = -0.104$$
$$h(3) = h_d(3)w_{Hn}(3) = (-0.075)(0.345) = -0.026$$
$$h(4) = h_d(4)w_{Hn}(4) = (0)(0.8145) = 0$$
$$h(5) = h_d(5)w_{Hn}(5) = (0.045)(0) = 0$$

Step 4: The transfer function of the filter is given by

The transfer function of the realizable filter is

$$H'(z) = z^{-\left(\frac{M-1}{2}\right)}H(z)$$
$$= z^{-5}\left[0.75 - 0.204z^{-1} - 0.204z^{1} - 0.104z^{-2} - 0.104z^{2} - 0.026z^{-3} - 0.026z^{3}\right]$$
$$= 0.75z^{-5} - 0.204z^{-6} - 0.204z^{-4} - 0.104z^{-7} - 0.104z^{-3} - 0.026z^{-8} - 0.026z^{-2}H$$

$$H(z) = h(0) + \sum_{n=1}^{\frac{M-1}{2}} h(n)\left(z^{-n} + z^{n}\right)$$
$$= h(0) + \sum_{n=1}^{5} h(n)\left[z^{-n} + z^{n}\right]$$
$$= 0.75 + h(1)\left[z^{-1} + z^{1}\right] + h(2)\left[z^{-2} + z^{2}\right] + h(3)\left[z^{-3} + z^{3}\right] + h(4)\left[z^{-4} + z^{4}\right] + h(5)\left[z^{-5} + z^{5}\right]$$
$$= 0.75 - 0.204z^{-1} - 0.204z^{1} - 0.104z^{-2} - 0.104z^{2} - 0.026z^{-3} - 0.026z^{3}$$

The causal filter coefficients are

$h(0) = h(1) = h(9) = h(10) = 0;$

$h(2) = h(8) = -0.026$

$h(3) = h(7) = -0.104$

$h(4) = h(6) = -0.204$

$h(5) = 0.75$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} a(n)\cos\omega n$$

$$a(0) = h\left(\frac{M-1}{2}\right) = h(5) = 0.75$$

$$a(n) = 2h\left[\frac{M-1}{2} - n\right] = 2h(5-n)$$

$$a(1) = 2h(5-1) = 2h(4) = -0.408$$
$$a(2) = 2h(5-2) = 2h(3) = -0.208$$
$$a(3) = 2h(5-3) = 2h(2) = -0.052$$
$$a(4) = 2h(5-4) = 2h(1) = 0$$
$$a(5) = 2h(5-5) = 2h(0) = 0$$

$$\bar{H}(e^{j\omega}) = 0.75 - 0.408\cos\omega - 0.208\cos 2\omega - 0.052\cos 3\omega$$

ω (in degrees)	0	1	2	3	4	5
$\bar{H}(e^{j\omega})$	0.082	0.0822	0.083	0.08433	0.08615	0.08848
$\left\lvert H(e^{j\omega})\right\rvert_{dB}$	-21.72	-21.70	-21.61	-21.480	-21.29	-21.11

b) Using Hamming window:

The hamming window sequence is given by

$$w_{Hm}(n) = \begin{cases} 0.54 + 0.46\cos\left(\frac{2\pi n}{M-1}\right); \; for -\left(\frac{M-1}{2}\right) \le n \le \left(\frac{M-1}{2}\right) \\ 0 \qquad otherwise \end{cases}$$

$$w_{Hm}(n) = \begin{cases} 0.54 + 0.46\cos\left(\frac{\pi n}{5}\right); \; for -5 \le n \le 5 \\ 0 \qquad otherwise \end{cases}$$

$$w_{Hm}(0) = 1$$
$$w_{Hm}(1) = w_{Hm}(-1) = 0.912$$
$$w_{Hm}(2) = w_{Hm}(-2) = 0.682$$
$$w_{Hm}(3) = w_{Hm}(-3) = 0.398$$
$$w_{Hm}(4) = w_{Hm}(-4) = 0.1678$$
$$w_{Hm}(5) = w_{Hm}(-5) = 0.08$$

The coefficients using hamming window sequence are,

$$h(n) = h_d(n) * w_{Hm}(n); \; -5 \le n \le 5$$
$$h(0) = h_d(0) * w_{Hm}(0) = 1 * 0.75 = 0.75$$
$$h(1) = h_d(1) * w_{Hm}(1) = (-0.225) * (0.912) = -0.2052$$
$$h(2) = h_d(2) * w_{Hm}(2) = (-0.159) * (0.682) = -0.1084$$
$$h(3) = h_d(3) * w_{Hm}(3) = (-0.075) * (0.398) = -0.03$$
$$h(4) = h_d(4) * w_{Hm}(4) = (0) * (0.16787) = 0$$
$$h(5) = h_d(5) * w_{Hm}(5) = (-0.045) * (0.08) = 0.0036$$

The transfer function of the filter is given by

$$H(z) = h(0) + \sum_{n=1}^{\frac{M-1}{2}} h(n)\left[z^{-n} + z^{n}\right]$$
$$= h(0) + \sum_{n=1}^{5} h(n)\left[z^{-n} + z^{n}\right]$$
$$= 0.75 + h(1)\left[z^{-1} + z^{1}\right] + h(2)\left[z^{-2} + z^{2}\right] + h(3)\left[z^{-3} + z^{3}\right] + h(4)\left[z^{-4} + z^{4}\right] + h(5)\left[z^{-5} + z^{5}\right]$$
$$= 0.75 - 0.2052z^{-1} - 0.2052z^{1} - 0.1084z^{-2} - 0.1084z^{2} - 0.03z^{-3} - 0.03z^{3} + 0.0036z^{-5} + 0.0036z^{5}$$

The transfer function of the realizable filter is

$$H'(z) = z^{-\left(\frac{M-1}{2}\right)} H(z)$$
$$= z^{-5}\left[0.75 - 0.2052z^{-1} - 0.2052z^{1} - 0.1084z^{-2} - 0.1084z^{2} - 0.03z^{-3} - 0.03z^{3} + 0.0036z^{-5} + 0.0036z^{5}\right]$$
$$H'(z) = 0.75z^{-5} - 0.2052z^{-6} - 0.2052z^{-4} - 0.1084z^{-7} - 0.1084z^{-3} - 0.03z^{-8} - 0.03z^{-2} + 0.0036z^{-10} + 0.0036$$

The filter coefficients of causal filter are

$h(0) = h(10) = 0.0036; h(1) = h(9) = 0; h(2) = h(8) = -0.03; h(3) = h(7) = -0.1084; h(4) = h(6) = -0.2052;$
$h(5) = 0.75$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} a(n)\cos\omega n$$

$$a(0) = h\left(\frac{M-1}{2}\right) = h(5) = 0.75$$
$$a(n) = 2h\left(\frac{M-1}{2} - n\right) = 2h(5-n)$$
$$a(1) = 2h(5-1) = 2h(4) = -0.4104$$
$$a(2) = 2h(5-2) = 2h(3) = -0.2168$$

$$a(3) = 2h(5-3) = 2h(2) = -0.06$$
$$a(4) = 2h(5-4) = 2h(1) = 0$$
$$a(5) = 2h(5-5) = 2h(0) = 0.0072$$

$$\bar{H}(e^{j\omega}) = 0.75 - 0.4104\cos\omega - 0.2168\cos 2\omega - 0.06\cos 3\omega + 0.0072\cos 5\omega$$

ω *(in degrees)*	**0**	**30**	**60**	**90**	**120**	**150**	**180**
$\bar{H}(e^{j\omega})$	**0.07**	**0.28**	**0.7168**	**0.9668**	**1**	**1.003**	**1.0108**
$\left\|H(e^{j\omega})\right\|_{dB}$	**-23.1**	**-11**	**-2.89**	**-0.29**	**0**	**0.028**	**0.093**

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For a FIR linear phase digital filter approximating the ideal frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{\pi}{6} \\ 0 & \frac{\pi}{6} \le |\omega| \le \pi \end{cases}$$

Determine the coefficients of a 5 tap filter using rectangular window.

Solution:

Given:

$$H_d(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \frac{\pi}{6} \\ 0 & \frac{\pi}{6} \le |\omega| \le \pi \end{cases}$$

Hence $\omega_C = \frac{\pi}{6}; N = 5.$

Step 1: To find filter coefficient.

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1.e^{j\omega n}\, d\omega$$

$$= \frac{\sin(n-\alpha)\omega_C}{\pi(n-\alpha)}; \qquad \because \alpha = 0$$

$$h_d(n) = \frac{\sin\frac{\pi}{6}n}{\pi n}; \quad -2 \le n \le 2$$

For n=0: $h_d(0) = \frac{\sin\frac{\pi}{6}(0)}{\pi(0)}$

$$= \frac{1}{6}\frac{\sin\frac{\pi n}{6}}{\frac{\pi n}{6}}$$

$$h_d(0) = \frac{1}{6} = 0.16 \qquad \therefore \lim_{\theta->0}\frac{\sin\theta}{\theta} = 1$$

$$for\ n = 1 \qquad h_d(1) = h_d(-1) = \frac{\sin\frac{\pi}{6}(1)}{\pi(1)} = \frac{0.5}{\pi} = 0.159$$

$$For\ n = 2; \qquad h_d(2) = h_d(-2) = \frac{\sin\frac{\pi(2)}{6}}{2\pi} = 0.1379$$

Step 2: Using Rectangular window:

$$w_R(n) = \begin{cases} 1 & for -\left(\frac{M-1}{2}\right) \le n \le \left(\frac{M-1}{2}\right) \\ 0 & for\ otherwise. \end{cases}$$

$$= 1 \quad for -2 \le n \le 2$$

$$w_R(0) = w_R(1) = w_R(2) = 1$$

Step 3: To find filter coefficients using rectangular window are

$$h(n) = h_d(n) * w_{Hn}(n) \quad for -2 \le n \le 2$$

$$h(0) = h_d(0) * w_{Hn}(0) = (0.16)*1 = 0.16$$

$$h(1) = h_d(1) * w_{Hn}(1) = 0.59*1 = 0.59$$

$$h(2) = h_d(2) * w_{Hn}(2) = 0.137*1 = 0.137$$

Step 4: The transfer function of the filter is given by

$$H(z) = h(0) + \sum_{n=1}^{\frac{M-1}{2}}\left[h(n)\left(z^n + z^{-n}\right)\right]$$

$$= 0.5 + \sum_{n=1}^{2} h(n)\left(z^n + z^{-n}\right)$$

$$= 0.16 + h(1)\left(z^1 + z^{-1}\right) + h(2)\left(z^2 + z^{-2}\right)$$

$$= 0.16 + 0.59z^1 + 0.59z^{-1} + 0.137z^{-2} + 0.137z^2$$

Step 5 : The transfer function of the realizable filter is

$$H'(z) = z^{-\frac{(M-1)}{2}} H(z)$$

$$= z^{-2}\left[0.16 + 0.59z^1 + 0.59z^{-1} + 0.137z^{-2} + 0.137z^2\right]$$

$$H'(z) = 0.16z^{-2} + 0.59z^{-1} + 0.59z^{-3} + 0.137z^{-4} + 0.137$$

The filter coefficients of causal filter are given by

$$h(0) = h(4) = 0.137, h(1) = h(3) = 0.59; h(2) = 0.16; h(5) = 0$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} a(n)\cos\omega n$$

$$a(0) = h\left(\frac{M-1}{2}\right) = h(2) = 0.32$$

$$a(n) = 2h\left(\frac{M-1}{2} - n\right) = 2h(2-n)$$

$$a(1) = 2h(2-1) = 2h(1) = 1.18$$

$$a(2) = 2h(2-2) = 2h(0) = 0.274$$

$$H(e^{j\omega}) = 0.32 + 1.18\cos\omega + 0.274\cos 2\omega$$

ω *(in degrees)*	0	30	60	90	120	150	180
$\bar{H}(e^{j\omega})$	1.774	1.47	0.773	0.046	-0.407	-0.564	-0.586

Design an ideal band pass filter with a frequency response

$$H_d(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4} \\ 0 & otherwise \end{cases}$$

Find the values of *h(n)* for N=11 using rectangular window.

Given:

$$H_d(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4} \\ 0 & otherwise \end{cases}$$

$$\omega_{C1} = \frac{\pi}{4} \text{ and } \omega_{C2} = \frac{3\pi}{4}$$

Step 1: Filter coefficients are,

$$h_d(n) = \frac{1}{\pi(n-\alpha)}\left[\sin\omega_{C2}(n-\alpha) - \sin\omega_{C1}(n-\alpha)\right] for\ n \ne \alpha$$

$$= \frac{1}{\pi n}\left[\sin\frac{3\pi n}{4} - \sin\frac{\pi n}{4}\right]$$

For n = 0;

$$h_d(0) = 0.5$$

$$h_d(1) = h_d(-1) = \frac{\sin\frac{3\pi(1)}{4} - \sin\frac{\pi(1)}{4}}{\pi(1)} = 0$$

$$h_d(2) = h_d(-2) = \frac{\sin\frac{3\pi(2)}{4} - \sin\frac{\pi(2)}{4}}{\pi(2)} = -0.3183$$

$$h_d(3) = h_d(-3) = \frac{\sin\frac{3\pi(3)}{4} - \sin\frac{\pi(3)}{4}}{\pi(3)} = 0$$

$$h_d(4) = h_d(-4) = \frac{\sin\frac{3\pi(4)}{4} - \sin\frac{\pi(4)}{4}}{\pi(4)} = 0$$

$$h_d(5) = h_d(-5) = \frac{\sin\frac{3\pi(5)}{4} - \sin\frac{\pi(5)}{4}}{\pi(5)} = 0$$

Step 2: Using rectangular window

$$w_R(n) = \begin{cases} 1 & for -\left(\frac{M-1}{2}\right) \le n \le \left(\frac{M-1}{2}\right) \\ 0 & for\ otherwise. \end{cases}$$

$$= 1 \quad for -2 \le n \le 2$$

$$w_R(0) = w_R(1) = w_R(2) = w_R(3) = w_R(4) = w_R(5) = 1$$

Step 3: Filter coefficients using rectangular window

$$h(n) = w_R(n) * h_d(n); \quad -5 \le n \le 5$$

$$h(0) = w_R(0) * h_d(0) = 1 * 0.5 = 0.5$$

$$h(1) = w_R(1) * h_d(1) = 1 * 0 = 0$$

$$h(2) = w_R(2) * h_d(2) = 1 * -0.3183 = -0.3183$$

$$h(3) = w_R(3) * h_d(3) = 1 * 0 = 0$$

$$h(4) = w_R(4) * h_d(4) = 1 * 0 = 0$$

$$h(5) = w_R(5) * h_d(5) = 1 * 0 = 0$$

Step 4: The transfer function of the filter is

$$H(z) = h(0) + \sum_{n=1}^{\frac{M-1}{2}} h(n)\left[z^{-n} + z^{n}\right]$$

$$= 0.5 - 0.3183z^{2} - 0.3183z^{-2}$$

Step 5: The transfer function of the realizable filter is

$$H'(z) = z^{-5}\left(0.5 - 0.3183z^{2} - 0.3183z^{-2}\right)$$

$$= 0.5z^{-5} - 0.3183z^{-3} - 0.3183z^{-7}$$

The filter coefficients of the causal filers are

$$h(0) = h(10) = h(9) = h(2) = h(8) = h(4) = h(6) = 0$$

$$h(3) = h(7) = -0.3183$$

$$h(5) = 0.5$$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} a(n)\cos\omega n$$

$$a(0) = h\left(\frac{M-1}{2}\right) = h(5) = 0.5$$

$$a(n) = 2h\left(\frac{M-1}{2} - n\right) = 2h(5-n)$$

$$a(1) = 2h(5-1) = 2h(4) = 0$$

$$a(2) = 2h(5-2) = 2h(3) = -0.6366$$

$$a(3) = 2h(5-3) = 2h(2) = 0$$

$$a(4) = 2h(5-4) = 2h(1) = 0$$

$$a(5) = 2h(5-5) = 2h(0) = 0$$

$$\bar{H}(e^{j\omega}) = 0.5 - 0.6366\cos 2\omega$$

ω *(in degrees)*	**0**	**30**	**60**	**90**	**120**	**150**	**180**
$\bar{H}(e^{j\omega})$	**-0.1366**	**0.1817**	**0.818**	**1.1366**	**0.818**	**0.1817**	**-0.1366**
$\lvert H(e^{j\omega})\rvert_{dB}$	**-17.3**	**-14.8**	**-1.74**	**1.11**	**-1.74**	**-14.8**	**-17.3**

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Design an ideal band Reject filter with a frequency response Find the values of *h(n)* for N=11 using rectangular window.

$$H(e^{j\omega}) = \begin{cases} 1 & for|\omega| \le \frac{\pi}{3}\ and\ |\omega| \le \frac{2\pi}{3} \\ 0 & otherwise \end{cases}$$

Solution:

Given:

$$H_d(e^{j\omega}) = \begin{cases} 1 & for|\omega| \le \frac{\pi}{3} \text{ and } |\omega| \le \frac{2\pi}{3} \\ 0 & otherwise \end{cases}$$

$$\omega_{C1} = \frac{\pi}{3} \text{ and } \omega_{C2} = \frac{2\pi}{3}$$

Step 1: Filter coefficients are,

$$h_d(n) = \frac{1}{\pi(n-\alpha)}\left[\sin \omega_{C1}(n-\alpha) - \sin \omega_{C2}(n-\alpha) + \sin(n-\alpha)\pi\right]$$

$$= \frac{1}{\pi n}\left[\sin\frac{\pi n}{3} - \sin\frac{2\pi n}{3} + \sin n\pi\right]$$

For n = 0;

$$h_d(0) = 0.667$$

$$h_d(1) = h_d(-1) = \frac{\sin\frac{\pi n}{3} - \sin\frac{2\pi n}{3} + \sin n\pi}{\pi(1)} = 0$$

$$h_d(2) = h_d(-2) = \frac{\sin\frac{\pi n}{3} - \sin\frac{2\pi n}{3} + \sin n\pi}{\pi(2)} = 0.2757$$

$$h_d(3) = h_d(-3) = \frac{\sin\frac{\pi n}{3} - \sin\frac{2\pi n}{3} + \sin n\pi}{\pi(3)} = 0$$

$$h_d(4) = h_d(-4) = \frac{\sin\frac{\pi n}{3} - \sin\frac{2\pi n}{3} + \sin n\pi}{\pi(4)} = -0.1378$$

$$h_d(5) = h_d(-5) = \frac{\sin\frac{\pi n}{3} - \sin\frac{2\pi n}{3} + \sin n\pi}{\pi(5)} = 0$$

Step 2: Using rectangular window

$$w_R(n) = \begin{cases} 1 & for -\left(\frac{M-1}{2}\right) \le n \le \left(\frac{M-1}{2}\right) \\ 0 & for\ otherwise. \end{cases}$$

$$= 1 \quad for -2 \le n \le 2$$

$$w_R(0) = w_R(1) = w_R(2) = w_R(3) = w_R(4) = w_R(5) = 1$$

Step 3: Filter coefficients using rectangular window

$h(n) = w_R(n) * h_d(n); \quad -5 \le n \le 5$

$h(0) = w_R(0) * h_d(0) = 1 * 0.667 = 0.667$

$h(1) = w_R(1) * h_d(1) = 1 * 0 = 0$

$h(2) = w_R(2) * h_d(2) = 1 * 0.2757 = 0.2757$

$h(3) = w_R(3) * h_d(3) = 1 * 0 = 0$

$h(4) = w_R(4) * h_d(4) = 1 * -0.1378 = -0.1378$

$h(5) = w_R(5) * h_d(5) = 1 * 0 = 0$

Step 4: The transfer function of the filter is

$$H(z) = h(0) + \sum_{n=1}^{\frac{M-1}{2}} h(n)\left[z^{-n} + z^{n}\right]$$

$$= 0.667 + 0.2757z^{2} + 0.2757z^{-2} - 0.1378z^{-4} - 0.1378z^{4}$$

Step 5: The transfer function of the realizable filter is

$$H'(z) = z^{-5}\left(0.667 + 0.2757z^{2} + 0.2757z^{-2} - 0.1378z^{-4} - 0.1378z^{4}\right)$$

$$= 0.667z^{-5} + 0.2757z^{-3} + 0.2757z^{-7} - 0.1378z^{-9} - 0.1378z^{-1}$$

The filter coefficients of the causal filers are

$h(0) = h(10) = h(2) = h(8) = h(4) = h(6) = 0$

$h(1) = h(9) = -0.1378$

$h(3) = h(7) = 0.2757$

$h(5) = 0.667$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{M-1}{2}} a(n)\cos\omega n$$

$$a(0) = h\left(\frac{M-1}{2}\right) = h(5) = 0.667$$

$$a(n) = 2h\left(\frac{M-1}{2} - n\right) = 2h(5-n)$$

$a(1) = 2h(5-1) = 2h(4) = 0$

$a(2) = 2h(5-2) = 2h(3) = 0.5514$

$a(3) = 2h(5-3) = 2h(2) = 0$

$a(4) = 2h(5-4) = 2h(1) = -0.2756$

$a(5) = 2h(5-5) = 2h(0) = 0$

$\bar{H}(e^{j\omega}) = 0.667 + 0.5514\cos 2\omega - 0.2756\cos 4\omega$

ω *(in degrees)*	**0**	**30**	**60**	**90**	**120**	**150**	**180**
$\bar{H}(e^{j\omega})$	**0.9428**	**1.08**	**0.526**	**16**	**0.529**	**1.08**	**0.9428**
$\left\|H(e^{j\omega})\right\|_{dB}$	**-0.5**	**0.67**	**-5.53**	**-15.9**	**-5.53**	**0.67**	**-0.5**

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Design a high pass filter using window, with a cut-off frequency of 1.2 radians/sec and N=9. [Nov/Dec-2016]

Solution:

Given

$\Omega_c = 1.2 radians / \sec$

$if \; T = 1 \sec$

$\omega_c = \Omega_c T = 1.2 \; radians$

The impulse response of a high pass filter with a cut off frequency ω_c is

$$h_d(\mathrm{n}) = \frac{-\sin\omega_c n}{\pi} \quad |n| \succ 0$$

$$1-\frac{\omega_c}{\pi} \qquad \text{for n=0}$$

$$\omega_c = 1.2$$

$$h_d(0) = 1 - \frac{1.2}{\pi} = 0.618$$

$$h_d(-1) = h_d(1) = \frac{-\sin 1.2}{\pi} = -0.2966$$

$$h_d(-2) = h_d(2) = \frac{-\sin 2.4}{2\pi} = -0.1075$$

$$h_d(-3) = h_d(3) = \frac{-\sin 3.6}{3\pi} = 0.0469$$

$$h_d(-4) = h_d(4) = \frac{-\sin 4.8}{4\pi} = 0.0719$$

Hamming window for $-4 \le n \le 4$ *is*

$$w_H(\mathrm{n}) = 0.54 + 0.46\cos\frac{2\pi n}{8} \quad \text{for -4} \le \text{n} \le 4$$

$$=0.54+0.46\cos\frac{\pi n}{4}$$

$$w_H(0) = 1$$

$$w_H(-1) = w_H(1) = 0.865$$

$$w_H(-2) = w_H(2) = 0.54$$

$$w_H(-3) = w_H(3) = 0.215$$

$$w_H(-4) = w_H(4) = 0.08$$

$$\mathrm{h(n)} = \mathrm{h}_d(\mathrm{n})\,\mathrm{w}_H(\mathrm{n})$$

$$h(0) = (0.618)(1) = 0.618$$

$$h(-1) = h(1) = (-0.2966)(0.865) = -0.256$$

$$h(-2) = h(2) = -0.058$$

$$h(-3) = h(3) = 0.01$$

$$h(-4) = h(4) = 0.0057$$

The casual filter coefficients are

$$h(0) = (0.618)(1) = 0.618$$

$$h(-1) = h(1) = (-0.2966)(0.865) = -0.256$$

$$h(-2) = h(2) = -0.058$$

$$h(-3) = h(3) = 0.01$$

$$h(-4) = h(4) = 0.0057$$

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