UNIT – IV FOURIER TRANSFORMS

4.1 FOURIER TRANSFORMS PAIR

1. **State Fourier integral theorem.**

   Solution :

   If $f(x)$ is piecewise continuous, differentiable and absolutely integrable in $(-\infty,\infty)$ then

   $$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\, e^{is(x-t)}\, dtds$$

2. **If F(s) is the Fourier transform of f (x), then show that** $F\{f(x-a)\}=e^{ias}F(s)$

   Solution :

   Given $F[f(x)]=F(s)$

   The Fourier Transform of *f(x)* is

   $$F[f(x)]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

   $$F[f(x-a)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-a)e^{isx}dx$$

   Let $x-a=t \Rightarrow dx=dt$

   $$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{is(t+a)}dt$$

   $$=e^{isa}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{ist}dt$$

   $$\boxed{F[f(x-a)]=e^{ias}F[f(x)]}$$

3. **State Convolution theorem in Fourier Transform.**

   Solution :

   The Fourier transform of the convolution of $f(x)$ and $g(x)$ is the product of their Fourier transforms .

   i.e. $F[f(x)*g(x)]=F[f(x)]F[g(x)]=F(s).G(s)$

4. **If** $F\{f(x)\}=F(s)$, **then find** $F\{e^{iax}f(x)\}$.

   Solution :

   $$F[f(x)]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx$$

   $$F\left[e^{iax}f(x)\right]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iax}f(x)e^{isx}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx+iax}dx$$

   $$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(s+a)x}dx$$

   $$\boxed{F\left[e^{iax}f(x)\right]=F(s+a)}$$

5. **State and prove the change of scale property of Fourier Transform.**

   Statement:

   If $F[f(x)]=F(s)$ then $F[f(ax)]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(ax)\, e^{isx}\, dx$

Solution :

Given $F[f(x)] = F(s)$

The Fourier Transform of *f(x)* is

$$F[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx$$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\ e^{isx}\ dx \quad ,$$

If a > 0 $\quad Put\ ax = t \ \Rightarrow\ adx = dt \Rightarrow dx = \frac{dt}{a} \quad when\ x = -\infty \Rightarrow t = -\infty\ \ and\ \ x = \infty \Rightarrow t = \infty$

$$F[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ e^{i\left(\frac{s}{a}\right)t}\ \frac{dt}{a}$$

$$F[f(ax)] = \frac{1}{a}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ e^{i\left(\frac{s}{a}\right)t} dt = \frac{1}{a} F\left(\frac{s}{a}\right). \text{ -(1)}$$

If a < 0 $\quad Put\ ax = t,\ \ adx = dt, dx = \frac{dt}{a}$

$when\ x = -\infty \Rightarrow t = \infty\ \ and\ \ x = \infty \Rightarrow t = -\infty$

$$\Rightarrow F[f(ax)] = \frac{-1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(t)\ e^{i\left(\frac{s}{a}\right)t}\ \frac{dt}{a} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\ e^{i\left(\frac{s}{a}\right)t}\ \frac{dt}{a} = \frac{1}{a} F\left(\frac{s}{a}\right). \quad \text{---(2)}$$

From (1) & (2) we get $F(f(ax)) = \frac{1}{|a|} F\left(\frac{s}{a}\right),\ a \neq 0$

6. **Find the Fourier Sine transform of** $\frac{1}{x}$**.**

Solution :

The Fourier Sine Transform of *f(x)* is

$$F_s[f(x)] = F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\, dx$$

$$F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin sx}{x} dx$$

$$F_s\left(\frac{1}{x}\right) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \frac{\sin t}{t} dt = \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right) = \sqrt{\frac{\pi}{2}} \qquad \because \int_0^{\infty} \frac{\sin mx}{x} dx = \frac{\pi}{2}$$

PART-B

1. **Find the Fourier transforms of** $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ **and hence evaluate** $\int_0^{\infty} \frac{\sin x}{x} dx.$ **Using Parseval's identity, prove that** $\int_0^{\infty} \frac{\sin^2 t}{t^2} dt = \frac{\pi}{2}$**.**

Solution: Given $f(x) = \begin{cases} 1, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform f(x) is

$$F\left[f(x)\right]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx\ .$$

$$F(s)=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} 0\,e^{isx}dx+\int_{-a}^{a} 1e^{isx}dx+\int_{a}^{\infty} 0\ e^{isx}dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}(\cos sx+i\sin sx)\,dx \qquad \because\ e^{isx}=\cos sx+i\sin sx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}\cos sx\ dx+i\int_{-a}^{a}\sin sx\,dx\right] \qquad \because\ \sin sx \text{ is an odd fn} \Rightarrow \int_{-a}^{a}\sin sx\,dx=0$$

$$=\frac{1}{\sqrt{2\pi}}2\int_{0}^{a}\cos sx\ dx$$

$$=\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[\frac{\sin sx}{s}\right]_0^a \qquad =\sqrt{\frac{2}{\pi}}\left[\frac{\sin as}{s}-0\right]$$

$$\boxed{F(s)=\sqrt{\frac{2}{\pi}}\left[\frac{\sin as}{s}\right]}$$

Deduction: 1

By inverse Fourier transform of *F(s)*.

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\sqrt{\frac{2}{\pi}}\frac{\sin as}{s}\right)e^{-isx}ds$$

$$=\frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)\left(\cos sx-i\sin sx\right)ds$$

$$=\frac{\sqrt{2}}{\sqrt{2}\sqrt{\pi}\sqrt{\pi}}\left[\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)\left(\cos sx\right)ds-i\int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)\left(\sin sx\right)ds\right]$$

$$f(x)=\frac{2}{\pi}\int_{0}^{\infty}\left(\frac{\sin as}{s}\right)\cos sx\,ds \qquad \because\left(\frac{\sin as}{s}\right)\sin sx \ \textit{is an odd fn} \Rightarrow \int_{-\infty}^{\infty}\left(\frac{\sin as}{s}\right)\sin sx\,dx=0$$

$$\int_{0}^{\infty}\left(\frac{\sin as}{s}\right)\cos sx\,ds=\frac{\pi}{2}f(x)$$

Put $x = 0$

$$\int_{0}^{\infty}\left(\frac{\sin as}{s}\right)\cos\left(0\right)ds=\frac{\pi}{2}f(0)$$

$$\int_{0}^{\infty}\left(\frac{\sin as}{s}\right)ds=\frac{\pi}{2}(1) \qquad \because\ f(x)=1\Rightarrow f(0)=1$$

Put $a=1$ and $s=x$ we get

$$\boxed{\therefore\int_{0}^{\infty}\left(\frac{\sin x}{x}\right)dx=\frac{\pi}{2}}$$

(ii) By Parseval's identity,

$$\int_{-\infty}^{\infty}\left[F(s)\right]^2dx=\int_{-\infty}^{\infty}\left[f(x)\right]^2ds$$

$$\int_{-\infty}^{\infty}\left[\sqrt{\frac{2}{\pi}}\frac{\sin sa}{s}\right]^2 ds=\int_{-a}^{a}(1)^2\,dx$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left[\frac{\sin sa}{s}\right]^2 ds=[x]_{-a}^{a}$$

$$\frac{2}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin sa}{s}\right)^2 ds=[a-(-a)]$$

$$\frac{2}{\pi}\cancel{2}\int_{0}^{\infty}\left(\frac{\sin sa}{s}\right)^2 ds=\cancel{2}a$$

$$\int_{0}^{\infty}\left(\frac{\sin sa}{s}\right)^2 ds=\frac{\cancel{2}\pi a}{\cancel{4}}=\frac{\pi a}{2}$$

Put $a=1$ & $s=t$ we get,

$$\boxed{\int_{0}^{\infty}\frac{\sin^2 t}{t^2}dt=\int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}}.$$

2. **Find the Fourier transform of** $f(x)=\begin{cases}x\,; & if\ |x|<a\\ 0\,; & if\ |x|>a\end{cases}$.

Solution: Given $f(x)=\begin{cases}x, & -a<x<a\\ 0, & \text{otherwise}\end{cases}$

The Fourier transform f(x) is

$$F[f(x)]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx\,.$$

$$F(s)=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a}0e^{isx}dx+\int_{-a}^{a}x\,e^{isx}dx+\int_{a}^{\infty}0\,e^{isx}dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}x(\cos sx+i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\cancel{\int_{-a}^{a}x\cos sx\,dx}+i\int_{-a}^{a}x\sin sx\,dx\right]\quad \because x\cos sx\ \text{is an}\ odd\ \text{fn}\ \therefore\int_{-a}^{a}x\cos sx\,dx=0$$

$$=i\frac{1}{\sqrt{2\pi}}2\int_{0}^{a}x\sin sx\,dx\qquad \because\ x\sin x\ \ is\ an\ even\ function\Rightarrow\int_{-a}^{a}x\sin sx\,dx=2\int_{0}^{a}x\sin sx\,dx$$

$$=i\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[(x)\left(\frac{-\cos sx}{s}\right)-(1)\left(\frac{-\sin sx}{s^2}\right)\right]_0^a$$

$$=i\sqrt{\frac{2}{\pi}}\left[-\frac{x\cos sx}{s}+\frac{\sin sx}{s^2}\right]_0^a$$

$$=i\sqrt{\frac{2}{\pi}}\left[\left(-\frac{a\cos sa}{s}+\frac{\sin sa}{s^2}\right)-(0)\right]$$

$$\boxed{F(s)=i\sqrt{\frac{2}{\pi}}\left[\left(\frac{\sin sa-as\cos sa}{s^2}\right)\right]}$$

3. **Find the Fourier transform of** $f(x)=\begin{cases} a-|x|\,;\ if\ |x|<a \\ 0\,;\qquad if\ |x|>a>0\end{cases}$ **is** $\sqrt{\frac{2}{\pi}}\left(\frac{1-\cos as}{s^2}\right)$**. Hence deduce that (i)** $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2 dt=\frac{\pi}{2}$. **(ii)** $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4 dt=\frac{\pi}{3}$.

Solution: Given $f(x)=\begin{cases} a-|x|, & -a<x<a \\ 0, & \text{otherwise}\end{cases}$

The Fourier transform $f(x)$ is

$$F\left[f(x)\right]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx\,.$$

$$F(s)=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a}0.e^{isx}dx+\int_{-a}^{a}\left(a-|x|\right)e^{isx}dx+\int_{a}^{\infty}0.e^{isx}dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\left(a-|x|\right)(\cos sx+i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}\left(a-|x|\right)\cos sx\ dx+\ i\int_{-a}^{a}\left(a-|x|\right)\sin sx\,dx\right]$$

$$\because\left(a-|x|\right)\sin sx\,\text{is an}\ odd\ \text{fn}\Rightarrow\int_{-a}^{a}\left(a-|x|\right)\sin sx\,dx=0$$

$$=\frac{1}{\sqrt{2\pi}}2\int_{0}^{a}\left(a-x\right)\cos sx\quad dx$$

$$=\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[\left(a-x\right)\left(\frac{\sin sx}{s}\right)-(-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^a$$

$$=\sqrt{\frac{2}{\pi}}\left[-\frac{\cos sx}{s^2}\right]_0^a$$

$$=\sqrt{\frac{2}{\pi}}\left[\left(-\frac{1}{s^2}\right)\left(\cos sa-\cos 0\right)\right]$$

$$\boxed{F(s)=\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos sa}{s^2}\right]}$$

$$\boxed{F(s)=2\sqrt{\frac{2}{\pi}}\left[\frac{\sin^2\left(\frac{as}{2}\right)}{s^2}\right]}\qquad \because \sin^2\theta=\frac{1-\cos 2\theta}{2}\Rightarrow 1-\cos\theta=2\sin^2\left(\frac{\theta}{2}\right)\ \text{here}\ \theta=\frac{as}{2}$$

Deduction: 1

By inverse Fourier transform of *F(s)*.

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(2\sqrt{\frac{2}{\pi}} \left[\frac{\sin^2\left(\frac{as}{2}\right)}{s^2} \right] \right) e^{-isx} ds$$

$$= \frac{2}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{as}{2}\right)}{s} \right)^2 (\cos sx - i \sin sx) ds$$

$$= \frac{2}{\pi} \left[\int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{as}{2}\right)}{s} \right)^2 (\cos sx) ds - i \int_{-\infty}^{\infty} \left(\frac{\sin\left(\frac{as}{2}\right)}{s} \right)^2 (\sin sx) ds \right]$$

$$f(x) = \frac{4}{\pi} \left[\int_{0}^{\infty} \left(\frac{\sin\left(\frac{as}{2}\right)}{s} \right)^2 (\cos sx) ds \right] \qquad \because \left(\frac{\sin\left(\frac{as}{2}\right)}{s} \right)^2 (\sin sx) \text{ is an odd function}$$

$$\int_{0}^{\infty} \left(\frac{\sin\left(\frac{as}{2}\right)}{s} \right)^2 \cos sx \, ds = \frac{\pi}{4} f(x)$$

Put $x = 0$

$$\int_{0}^{\infty} \left(\frac{\sin\left(\frac{as}{2}\right)}{s} \right)^2 (\cos 0) ds = \frac{\pi}{4} f(0)$$

$$\int_{0}^{\infty} \left(\frac{\sin\left(\frac{as}{2}\right)}{s} \right)^2 ds = \frac{\pi a}{4} \qquad \because f(x) = a - |x| \Rightarrow f(0) = a$$

Put a=1 and s=t get

$$\int_{0}^{\infty} \left(\frac{\sin\left(\frac{s}{2}\right)}{s} \right)^2 ds = \frac{\pi}{4} \qquad put\ \frac{s}{2} = t \Rightarrow \frac{ds}{2} = dt$$

$$\int_{0}^{\infty} \left(\frac{\sin t}{2t} \right)^2 2dt = \frac{\pi}{4}$$

$$\boxed{\therefore \int_{0}^{\infty} \left(\frac{\sin t}{t} \right)^2 dt = \frac{\pi}{2}}$$

(ii) By Parseval’s identity,

$$\int_{-\infty}^{\infty}[F(s)]^2\,dx=\int_{-\infty}^{\infty}[f(x)]^2\,ds$$

$$\int_{-\infty}^{\infty}\left[2\sqrt{\frac{2}{\pi}}\left[\frac{\sin^2\left(\frac{as}{2}\right)}{s^2}\right]\right]^2 ds=\int_{-a}^{a}\left(a-|x|\right)^2 dx$$

$$\frac{8}{\pi}2\int_{0}^{\infty}\left[\frac{\sin\left(\frac{as}{2}\right)}{s}\right]^4 ds=2\int_0^a\left(a-x\right)^2 dx \quad \because \left(a-|x|\right)^2 \text{ and } \frac{\sin^2\left(\frac{as}{2}\right)}{s^2} \text{ are even functions}$$

$$\frac{8}{\pi}\int_{0}^{\infty}\left[\frac{\sin\left(\frac{as}{2}\right)}{s}\right]^4 ds=\int_0^a\left(a-x\right)^2 dx$$

$$\frac{8}{\pi}\int_{0}^{\infty}\left[\frac{\sin\left(\frac{as}{2}\right)}{s}\right]^4 ds=\left[\frac{\left(a-x\right)^3}{-3}\right]_0^a$$

$$\frac{8}{\pi}\int_{0}^{\infty}\left[\frac{\sin\left(\frac{as}{2}\right)}{s}\right]^4 ds=\left[(0)-\left(\frac{-a^3}{3}\right)\right]$$

$$\int_{0}^{\infty}\left[\frac{\sin\left(\frac{as}{2}\right)}{s}\right]^4 ds=\frac{a^3\pi}{3\times 8}$$

Put $a=1$ & $s=t$ we get,

$$\int_{0}^{\infty}\left[\frac{\sin\left(\frac{s}{2}\right)}{s}\right]^4 ds=\frac{\pi}{24} \qquad\qquad put\,\frac{s}{2}=t\Rightarrow\frac{ds}{2}=dt$$

$$\int_{0}^{\infty}\left[\frac{\sin t}{2t}\right]^4 2dt=\frac{\pi}{24}$$

$$\boxed{\int_{0}^{\infty}\left[\frac{\sin t}{t}\right]^4 dt=\frac{\pi}{3}}.$$

4. **Find the Fourier transform of** $f(x)=\begin{cases}1-|x|, & |x|<1\\ 0, & |x|\geq 1\end{cases}$ **and hence find the value of**

(i) $\int_0^\infty \frac{\sin^2 t}{t^2}\,dt$. **(ii)** $\int_0^\infty \frac{\sin^4 t}{t^4}\,dt$.

Solution:

Hint : In the previous problem *a=1*.

5\. **Find the Fourier transform of** $f(x)=\begin{cases} a^2-x^2, & |x|\le a \\ 0, & |x|\ge a \end{cases}$ **and hence evaluate**

(i) $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)dt=\frac{\pi}{4}$ **(ii)** $\int_0^\infty \left(\frac{\sin t - t\cos t}{t^3}\right)^2 dt=\frac{\pi}{15}$

Solution: Given $f(x)=\begin{cases} a^2-x^2, & -a<x<a \\ 0, & \text{otherwise} \end{cases}$

The Fourier transform of f(x) is

$$F\left[f(x)\right]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}dx\,.$$

$$F(s)=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-a} 0\,e^{isx}dx+\int_{-a}^{a}\left(a^2-x^2\right)e^{isx}dx+\int_{a}^{\infty}0\;e^{isx}dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-a}^{a}\left(a^2-x^2\right)(\cos sx+i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{a}\left(a^2-x^2\right)\cos sx\;dx+i\int_{-a}^{a}\left(a^2-x^2\right)\sin sx\,dx\right]$$

$$\because \left(a^2-x^2\right)\sin sx \text{ is an } odd \text{ fn} \therefore \int_{-a}^{a}\left(a^2-x^2\right)\sin sx\,dx=0$$

$$=\frac{1}{\sqrt{2\pi}}2\int_0^a\left(a^2-x^2\right)\cos sx\,dx$$

$$=\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[\left(a^2-x^2\right)\left(\frac{\sin sx}{s}\right)-(-2x)\left(\frac{-\cos sx}{s^2}\right)+(-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^a$$

$$=-2\sqrt{\frac{2}{\pi}}\left[\frac{x\cos sx}{s^2}-\frac{\sin sx}{s^3}\right]_0^a$$

$$=-2\sqrt{\frac{2}{\pi}}\left[\left(\frac{a\cos sa}{s^2}-\frac{\sin sa}{s^3}\right)-(0)\right]$$

$$=-2\sqrt{\frac{2}{\pi}}\left[\frac{as\cos sa-\sin sa}{s^3}\right]$$

$$\boxed{F(s)=2\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa-as\cos sa}{s^3}\right]}$$

Deduction: 1

By inverse Fourier transform of *F(s)*.

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(2\sqrt{\frac{2}{\pi}}\left[\frac{\sin sa-as\cos sa}{s^3}\right]\right)e^{-isx}ds$$

$$= \frac{2}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \left[\frac{\sin sa - as\cos sa}{s^3} \right] (\cos sx - i\sin sx)\, ds$$

$$= \frac{2}{\pi} \left[\int_{-\infty}^{\infty} \left[\frac{\sin sa - as\cos sa}{s^3} \right] (\cos sx)\, ds - i \int_{-\infty}^{\infty} \left[\frac{\sin sa - as\cos sa}{s^3} \right] (\sin sx)\, ds \right]$$

$$f(x) = \frac{4}{\pi} \int_{0}^{\infty} \left[\frac{\sin sa - as\cos sa}{s^3} \right] \cos sx\, ds \qquad \because \left[\frac{\sin sa - as\cos sa}{s^3} \right] (\sin sx) \text{ is an odd function}$$

$$\int_{0}^{\infty} \left[\frac{\sin sa - as\cos sa}{s^3} \right] \cos sx\, ds = \frac{\pi}{4} f(x)$$

Put x=0

$$\int_{0}^{\infty} \left[\frac{\sin sa - as\cos sa}{s^3} \right] (\cos 0)\, ds = \frac{\pi}{4} f(0)$$

$$\int_{0}^{\infty} \left[\frac{\sin sa - as\cos sa}{s^3} \right] ds = \frac{\pi a^2}{4} \qquad \because f(x) = a^2 - x^2 \Rightarrow f(0) = a^2$$

Put a=1 and s=t get

$$\boxed{\int_{0}^{\infty} \left[\frac{\sin t - t\cos t}{t^3} \right] dt = \frac{\pi}{4}}$$

(ii) By Parseval's identity,

$$\int_{-\infty}^{\infty} [F(s)]^2\, ds = \int_{-\infty}^{\infty} [f(x)]^2 dx$$

$$\int_{-\infty}^{\infty} \left[2\sqrt{\frac{2}{\pi}} \left[\frac{\sin sa - as\cos sa}{s^3} \right] \right]^2 ds = \int_{-a}^{a} \left(a^2 - x^2\right)^2 dx$$

$$\frac{8}{\pi} 2\int_{0}^{\infty} \left[\frac{\sin sa - as\cos sa}{s^3} \right]^2 ds = 2\int_{0}^{a} \left(a^4 - 2a^2x^2 + x^4\right) dx$$

$$\because \left(a^2 - x^2\right)^2 \text{ and} \left[\frac{\sin sa - as\cos sa}{s^3} \right]^2 \text{ are even functions}$$

$$\frac{8}{\pi} \int_{0}^{\infty} \left[\frac{\sin sa - as\cos sa}{s^3} \right]^2 ds = \left(a^4 x - \frac{2a^2x^3}{3} + \frac{x^5}{5} \right)_0^a$$

$$\frac{8}{\pi} \int_{0}^{\infty} \left[\frac{\sin sa - as\cos sa}{s^3} \right]^2 ds = \left(a^5 - \frac{2a^5}{3} + \frac{a^5}{5} \right)$$

$$\frac{8}{\pi} \int_{0}^{\infty} \left[\frac{\sin sa - as\cos sa}{s^3} \right]^2 ds = \left(\frac{15a^5 - 10a^5 + 3a^5}{15} \right)$$

$$\int_{0}^{\infty} \left[\frac{\sin sa - as\cos sa}{s^3} \right]^2 ds = \left(\frac{8a^5}{15} \right) \times \frac{\pi}{8}$$

Put a =*1* & s=t we get,

$$\boxed{\int_{0}^{\infty} \left[\frac{\sin t - t\cos t}{t^3} \right]^2 dt = \frac{\pi}{15}}$$

6. **Find the Fourier transform of** $f(x)=\begin{cases}1-x^2\ ;\ if\ |x|<1\\ 0\ ;\quad if\ |x|\geq 1\end{cases}$.

Hence show that $(i)\int_0^\infty\left[\frac{\sin s-s\cos s}{s^2}\right]\cos\left(\frac{s}{2}\right)ds=\frac{3\pi}{16}$ and $(ii)\int_0^\infty\frac{(x\cos x-\sin x)^2}{x^6}dx=\frac{\pi}{15}$

Solution: Given $f(x)=\begin{cases}1-x^2, & -1<x<1\\ 0, & \text{otherwise}\end{cases}$

The Fourier transform f(x) is

$$F[f(x)]=F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{isx}dx\ .$$

$$F(s)=\frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-1}0\,e^{isx}dx+\int_{-1}^{1}\left(1-x^2\right)e^{isx}dx+\int_{1}^{\infty}0\ e^{isx}dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}\left(1-x^2\right)(\cos sx+i\sin sx)\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{1}\left(1-x^2\right)\cos sx\ dx+i\int_{-1}^{1}\left(1-x^2\right)\sin sx\,dx\right]$$

$$\because\left(1-x^2\right)\sin sx \text{ is an } odd \text{ fn}\therefore\int_{-1}^{1}\left(1-x^2\right)\sin sx\,dx=0$$

$$=\frac{1}{\sqrt{2\pi}}2\int_{0}^{1}\left(1-x^2\right)\cos sx\,dx$$

$$=\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}\sqrt{\pi}}\left[\left(1-x^2\right)\left(\frac{\sin sx}{s}\right)-(-2x)\left(\frac{-\cos sx}{s^2}\right)+(-2)\left(\frac{-\sin sx}{s^3}\right)\right]_0^1$$

$$=-2\sqrt{\frac{2}{\pi}}\left[\frac{x\cos sx}{s^2}-\frac{\sin sx}{s^3}\right]_0^1$$

$$=-2\sqrt{\frac{2}{\pi}}\left[\left(\frac{\cos s}{s^2}-\frac{\sin s}{s^3}\right)-(0)\right]$$

$$=-2\sqrt{\frac{2}{\pi}}\left[\frac{s\cos s-\sin s}{s^3}\right]$$

$$\boxed{F(s)=2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s-s\cos s}{s^3}\right]}$$

Deduction: 1

By inverse Fourier transform of *F(s)*.

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(s)e^{-isx}ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s-s\cos s}{s^3}\right]\right)e^{-isx}ds$$

$$=\frac{2}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left[\frac{\sin s-s\cos s}{s^3}\right](\cos sx-i\sin sx)\,ds$$

$$= \frac{2}{\pi}\left[\int_{-\infty}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right](\cos sx)\,ds - i\int_{-\infty}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right](\sin sx)\,ds\right]$$

$$f(x) = \frac{4}{\pi}\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]\cos sx\,ds \qquad \because \left[\frac{\sin s - s\cos s}{s^3}\right](\sin sx) \text{ is an odd function}$$

$$\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]\cos sx\,ds = \frac{\pi}{4}f(x)$$

Put $x = \frac{1}{2}$

$$\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]\cos\left(\frac{s}{2}\right)ds = \frac{\pi}{4}f\left(\frac{1}{2}\right)$$

$$\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]\cos\left(\frac{s}{2}\right)ds = \frac{\pi}{4}\times\frac{3}{4} \qquad \because f(x) = 1 - x^2 \Rightarrow f\left(\frac{1}{2}\right) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\boxed{\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]\cos\left(\frac{s}{2}\right)ds = \frac{3\pi}{16}}$$

(ii) **By Parseval's identity,**

$$\int_{-\infty}^{\infty}[F(s)]^2\,ds = \int_{-\infty}^{\infty}[f(x)]^2\,dx$$

$$\int_{-\infty}^{\infty}\left[2\sqrt{\frac{2}{\pi}}\left[\frac{\sin s - s\cos s}{s^3}\right]\right]^2 ds = \int_{-1}^{1}\left(1 - x^2\right)^2 dx$$

$$\frac{8}{\pi}2\int_{0}^{\infty}\left[\frac{\sin sa - as\cos sa}{s^3}\right]^2 ds = 2\int_{0}^{1}\left(1 - 2x^2 + x^4\right)dx$$

$$\because \left(1 - x^2\right)^2 \text{ and } \left[\frac{\sin s - s\cos s}{s^3}\right]^2 \text{ are an even functions}$$

$$\frac{8}{\pi}\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds = \left(x - \frac{2x^3}{3} + \frac{x^5}{5}\right)_0^1$$

$$\frac{8}{\pi}\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds = \left(1 - \frac{2}{3} + \frac{1}{5}\right)$$

$$\frac{8}{\pi}\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds = \left(\frac{15 - 10 + 3}{15}\right)$$

$$\int_{0}^{\infty}\left[\frac{\sin s - s\cos s}{s^3}\right]^2 ds = \left(\frac{8}{15}\right)\times\frac{\pi}{8}$$

Put $s = t$ we get,

$$\boxed{\int_{0}^{\infty}\frac{(\sin t - t\cos t)^2}{t^6}\,dx = \frac{\pi}{15}}$$