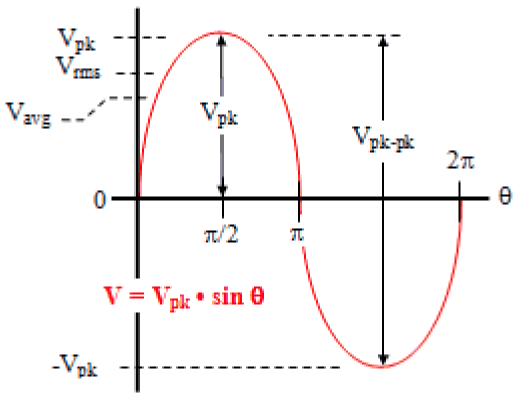
Average value, RMS Value, Form factor and Peak factor for different waveforms:

Sinusoidal wave:

A sinewave is defined by the trigonometric sine function. When plotted as voltage (V) as a function of phase (θ), it looks similar to the figure to the below. The waveform repeats every 2p radians (360°), and is symmetrical about the voltage axis (when no DC offset is present). Voltage and current exhibiting cyclic behavior is referred to as alternating; i.e., alternating current (AC). One full cycle is shown here. The basic equation for a sinewave is as follows:

$$V(\theta) = V_{pk} \cdot \sin(\theta)$$

There are a number of ways in which the amplitude of a sinewave is referenced, usually as peak voltage (V_{pk} or V_p), peak-to-peak voltage (V_{pp} or $V_{p\text{-}p}$ or V_{pkpk} or $V_{pk\text{-}pk}$), average voltage (V_{av} or V_{avg}), and root-mean-square voltage (V_{rms}). Peak voltage and peak-to-peak voltage are apparent by looking at the above plot. Root-mean-square and average voltage are not so apparent.



Average Voltage (V_{avg})

As the name implies, V_{avg} is calculated by taking the average of the voltage in an appropriately chosen interval. In the case of symmetrical waveforms like the sinewave, a quarter cycle faithfully represents all four quarter cycles of the waveform. Therefore, it is acceptable to choose the first quarter cycle, which goes from 0 radians (0°) through p/2 radians (90°).

As with the V_{rms} formula, a full derivation for the V_{avg} formula is given here as well.

$$V_{avg} = \frac{1}{\pi/2} \cdot \int_0^{\pi/2} V_{pk} \cdot \sin\theta \cdot d\theta = \frac{2}{\pi} \cdot V_{pk} \cdot -\cos\theta \Big|_0^{\pi/2}$$

$$= \frac{-2}{\pi} \cdot V_{pk} \cdot \left(\cos\frac{\pi}{2} - \cos 0\right) = \frac{-2}{\pi} \cdot V_{pk} \cdot (0-1) = \frac{2}{\pi} \cdot V_{pk}$$

$$\text{So,}\quad V_{avg} = \frac{2}{\pi}\cdot V_{pk} \approx 0.636\, V_{pk},$$

Root-Mean-Square Voltage (V_{rms})

As the name implies, V_{rms} is calculated by taking the square root of the mean average of the square of the voltage in an appropriately chosen interval. In the case of symmetrical waveforms like the sine wave, a quarter cycle faithfully represents all four quarter cycles of the waveform. Therefore, it is acceptable to choose the first quarter cycle, which goes from 0 radians (0°) through p/2 radians (90°).

V_{rms} is the value indicated by the vast majority of AC voltmeters. It is the value that, when applied across a resistance, produces that same amount of heat that a direct current (DC) voltage of the same magnitude would produce. For example, 1 V applied across a 1 Ω resistor produces 1 W of heat. A 1 V_{rms} sine wave applied across a 1 Ω resistor also produces 1 W of heat. That 1 V_{rms} sine wave has a peak voltage of √2 V (≈1.414 V), and a peak-to-peak voltage of 2√2 V (≈2.828 V).

Since finding a full derivation of the formulas for root-mean-square (V_{rms}) voltage is difficult, it is done here for you.

$$V_{rms\,(sinewave)} = \sqrt{\frac{1}{\pi/2}\int_0^{\pi/2}\left(V_{pk}\sin\theta\right)^2\cdot d\theta} = \sqrt{\frac{2\cdot V_{pk}^2}{\pi}\left(\frac{\theta}{2}-\frac{1}{4}\sin(2\cdot\theta)\right)\Big|_0^{\pi/2}}$$

$$= \frac{\sqrt{2}\cdot V_{pk}}{\sqrt{\pi}}\sqrt{\left(\frac{\pi/2}{2}-\frac{1}{4}\sin(\pi)\right)-\left(\frac{0}{2}-\frac{1}{4}\sin(0)\right)}$$

$$= \frac{\sqrt{2}\cdot V_{pk}}{\sqrt{\pi}}\sqrt{\left(\frac{\pi}{4}-0\right)-(0-0)} = \frac{\sqrt{2}\cdot V_{pk}}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}}\cdot V_{pk}$$

$$\text{So,}\quad V_{rms} = \frac{1}{\sqrt{2}}\cdot V_{pk} \approx 0.707\, V_{pk},$$

Form factor:

Two alternating periodic waveforms of the same amplitude and frequency may look different depending upon their wave shape/form and then their average & RMS values will be different. In order to compare such different waveforms of the same frequency and amplitude but of different wave shape a parameter called Form factor is defined as the ratio of it’s RMS and Average values.

For a sinusoidal signal of peak voltage V_m it is given by :

Form factor of a sinusoidal signal $= V_{rms} / V_{av}$
$= 0.707\, V_m / 0.637\, V_m = 1.11$

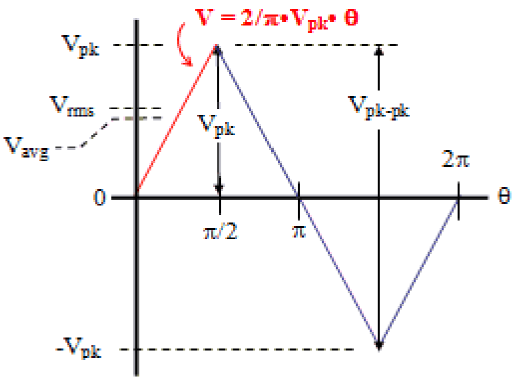
Peak Factor (Or Crest factor): Is defined as the ratio of maximum value to the R.M.S value of an alternating quantity.

Peak factor of a sinusoidal signal$=V_{max}/V_{rms}$

$=V_{max}/(0.707\ V_m)$

$=1.414$

Triangular wave:



When plotted as voltage (V) as a function of phase (θ), a triangle wave looks similar to the figure to the above. The waveform repeats every 2π radians (360°), and is symmetrical about the voltage axis (when no DC offset is present). Voltage and current exhibiting cyclic behavior is referred to as alternating; i.e., alternating current (AC). One full cycle is shown here. The basic equation for a triangle wave is as follows:

$$V = \frac{2}{\pi} \cdot V_{pk} \cdot \theta \quad \text{for } 0 \le \theta < \pi/2$$

There are a number of ways in which the amplitude of a triangle wave is referenced, usually as peak voltage (V_{pk} or V_p), peak-to-peak voltage (V_{pp} or V_{p-p} or V_{pkpk} or V_{pk-pk}), average voltage (V_{av} or V_{avg}), and root-mean-square voltage (V_{rms}). Peak voltage and peak-to-peak voltage are apparent by looking at the above plot. Root-mean-square and average voltage are not so apparent.

Average Voltage (V_{avg})

As the name implies, V_{avg} is calculated by taking the average of the voltage in an appropriately chosen interval. In the case of symmetrical waveforms like the triangle wave, a quarter cycle faithfully represents all four quarter cycles of the waveform. Therefore, it is acceptable to choose the first quarter cycle, which goes from 0 radians (0°) through π/2 radians (90°).

As with the V_{rms} formula, a full derivation for the V_{avg} formula is given here as well.

$$V_{avg} = \frac{1}{\pi/2}\int_0^{\pi/2} \frac{2}{\pi} \cdot V_{pk} \cdot \theta \cdot d\theta = \frac{2}{\pi} \cdot \frac{2}{\pi} \cdot V_{pk} \cdot \frac{1}{2} \cdot \theta^2 \Big|_0^{\pi/2}$$

$$= \frac{4}{\pi^2} \cdot V_{pk} \cdot \frac{1}{2} \cdot \left(\frac{\pi^2}{4} - 0\right) = \frac{1}{2} \cdot V_{pk}$$

$$V_{avg} = \frac{1}{2} \cdot V_{pk}$$

$\approx 0.5\ V_{pk}$

Root-Mean-Square Voltage (V_{rms})

As the name implies, V_{rms} is calculated by taking the square root of the mean average of the square of the voltage in an appropriately chosen interval. In the case of symmetrical waveforms like the triangle wave, a quarter cycle faithfully represents all four quarter cycles of the waveform. Therefore, it is acceptable to choose the first quarter cycle, which goes from 0 radians (0°) through π/2 radians (90°).

V_{rms} is the value indicated by the vast majority of AC voltmeters. It is the value that, when applied across a resistance, produces that same amount of heat that a direct current (DC) voltage of the same magnitude would produce. For example, 1 V applied across a 1 Ω resistor produces 1 W of heat. A 1 V_{rms} triangle wave applied across a 1 Ω resistor also produces 1 W of heat. That 1 V_{rms} triangle wave has a peak voltage of √3 V (≈1.732 V), and a peak-to-peak voltage of 2√3 V (≈3.464 V).

Since finding a full derivation of the formulas for root-mean-square (V_{rms}) voltage is difficult, it is done here for you.

$$V_{rms} = \sqrt{\frac{1}{\pi/2}\cdot\int_0^{\pi/2}\left(\frac{2}{\pi}\cdot V_{pk}\cdot\theta\right)^2 d\theta} = \sqrt{\frac{2}{\pi}\cdot\frac{4}{\pi^2}\cdot V_{pk}^2\cdot\int_0^{\pi/2}\theta^2 d\theta} = \sqrt{\frac{8}{\pi^3}\cdot V_{pk}^2\cdot\int_0^{\pi/2}\theta^2 d\theta}$$

$$= \sqrt{\frac{8}{\pi^3}\cdot V_{pk}^2\cdot\frac{1}{3}\cdot\theta^3\Big|_0^{\pi/2}} = \sqrt{\frac{8}{3\cdot\pi^3}\cdot V_{pk}^2\cdot\left[\left(\frac{\pi}{2}\right)^3 - 0\right]}$$

$$= \sqrt{\frac{8}{3\cdot\pi^3}\cdot V_{pk}^2\cdot\frac{\pi^3}{2^3}} = \sqrt{\frac{1}{3}\cdot V_{pk}^2} = \frac{1}{\sqrt{3}}\cdot V_{pk}$$

So, $V_{rms} = \frac{1}{\sqrt{3}}\cdot V_{pk} \approx 0.577\ V_{pk}$

Form factor:

Form factor of a triangular signal = V_{rms} /V_{av}

=.577Vpk/.5Vpk

=1.15

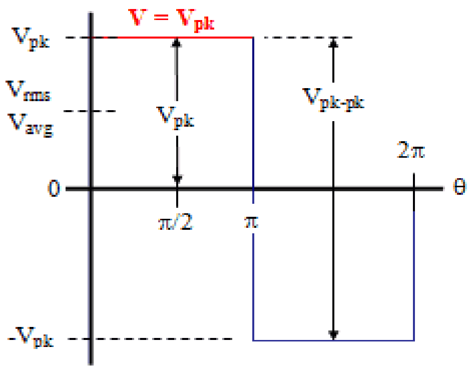
Peak Factor (Or Crest factor): Is defined as the ratio of maximum value to the R.M.S value

Peak factor of a triangular signal=V_{pk}/V_{rms}

=Vpk/.577Vpk

=1.732

Square wave:



When plotted as voltage (V) as a function of phase (θ), a square wave looks similar to the figure to the above. The waveform repeats every 2π radians (360°), and is symmetrical about the voltage axis (when no DC offset is present). Voltage and current exhibiting cyclic behavior is referred to as alternating; i.e., alternating current (AC). One full cycle is shown here.

The basic equation for a square wave is as follows:

$$V_{\text{one complete cycle}} = \begin{vmatrix} 1, & \text{for } 0 \le \theta < \pi \\ -1, & \text{for } \pi \le \theta < 2\pi \end{vmatrix}$$

There are a number of ways in which the amplitude of a square wave is referenced, usually as peak voltage (V_{pk} or V_p), peak-to-peak voltage (V_{pp} or $V_{p\text{-}p}$ or V_{pkpk} or $V_{pk\text{-}pk}$), average voltage (V_{av} or V_{avg}), and root-mean-square voltage (V_{rms}). Peak voltage and peak-to-peak voltage are apparent by looking at the above plot. Root-mean-square and average voltage are not so apparent.

Average Voltage (V_{avg})

As the name implies, V_{avg} is calculated by taking the average of the voltage in an appropriately chosen interval. In the case of symmetrical waveforms like the square wave, a quarter cycle faithfully represents all four quarter cycles of the waveform. Therefore, it is acceptable to choose the first quarter cycle, which goes from 0 radians (0°) through π/2 radians (90°).

As with the V_{rms} formula, a full derivation for the V_{avg} formula is given here as well.

$$\text{Vavg} = \frac{1}{\pi/2} \cdot \int_0^{\pi/2} V_{pk} \cdot d\theta = \frac{2}{\pi} \cdot V_{pk} \cdot \theta \Big|_0^{\pi/2}$$

$$= \frac{2}{\pi} \cdot V_{pk} \cdot \left(\frac{\pi}{2} - 0\right) = V_{pk}$$

So, $V_{avg} = V_{pk}$

Root-Mean-Square Voltage (V_{rms})

As the name implies, V_{rms} is calculated by taking the square root of the mean average of the square of the voltage in an appropriately chosen interval. In the case of symmetrical waveforms like the square wave, a quarter cycle faithfully represents all four quarter cycles of the waveform. Therefore, it is acceptable to choose the first quarter cycle, which goes from 0 radians (0°) through π/2 radians (90°).

V_{rms} is the value indicated by the vast majority of AC voltmeters. It is the value that, when applied across a resistance, produces that same amount of heat that a direct current (DC) voltage of the same magnitude would produce. For example, 1 V applied across a 1 Ω resistor produces 1 W of heat. A 1 V_{rms} square wave applied across a 1 Ω resistor also produces 1 W of heat. That 1 V_{rms} square wave has a peak voltage of 1 V, and a peak-to-peak voltage of 2 V.

Since finding a full derivation of the formulas for root-mean-square (V_{rms}) voltage is difficult, it is done here for you.

$$V_{rms} = \sqrt{\frac{1}{\pi/2} \cdot \int_0^{\pi/2} V_{pk}^2 \cdot d\theta} = \sqrt{\frac{2}{\pi} \cdot V_{pk}^2 \cdot \theta \Big|_0^{\pi/2}}$$
$$= \sqrt{\frac{2}{\pi} \cdot V_{pk}^2 \cdot \left(\frac{\pi}{2} - 0\right)} = \sqrt{V_{pk}^2} = V_{pk}$$

So, $V_{rms} = V_{pk}$

Form factor:

Form factor of a triangular signal = V_{rms} / V_{av}
=Vpk/Vpk
=1

Peak Factor (Or Crest factor):

Peak factor of a triangular signal=V_{pk}/V_{rms}
=Vpk/Vpk

J notation:

The mathematics used in Electrical Engineering to add together resistances, currents or DC voltages use what are called “real numbers” either as integers or as fractions.But real numbers are not the only kind of numbers we need to use especially when dealing with frequency dependent sinusoidal sources and vectors. As well as using normal or real numbers, Complex Numbers were introduced to allow complex equations to be solved with numbers that are the square roots of negative numbers, √-1.

In electrical engineering this type of number is called an “imaginary number” and to distinguish an imaginary number from a real number the letter “ j ” known commonly in electrical engineering as the j-operator, is used. The letter j is placed in front of a real number to signify its imaginary number operation.

Examples of imaginary numbers are: j3, j12, j100 etc. Then a complex number consists of two distinct but very much related parts, a “ Real Number ” plus an “ Imaginary Number ”.Complex Numbers represent points in a two dimensional complex or s-plane that are referenced to two distinct axes. The horizontal axis is called the “real axis” while the vertical axis is called the “imaginary axis”. The real and imaginary parts of a complex number are abbreviated as Re(z) and Im(z), respectively.

Complex numbers that are made up of real (the active component) and imaginary (the reactive component) numbers can be added, subtracted and used in exactly the same way as elementary algebra is used to analyse dc circuitsThe rules and laws used in mathematics for the addition or subtraction of imaginary numbers are the same as for real numbers, j2 + j4 = j6 etc. The only difference is in multiplication because two imaginary numbers multiplied together becomes a negative real number. Real numbers can also be thought of as a complex number but with a zero imaginary part labelled j0.

The j-operator has a value exactly equal to √-1, so successive multiplication of “ j “, (j x j) will result in j having the following values of, -1, -j and +1. As the j-operator is commonly used to indicate the anticlockwise rotation of a vector, each successive multiplication or power of “ j “, j^2, j^3 etc, will force the vector to rotate through an angle of 90^o anticlockwise as shown below. Likewise, if the multiplication of the vector results in a -j operator then the phase shift will be -90°, i.e. a clockwise rotation.

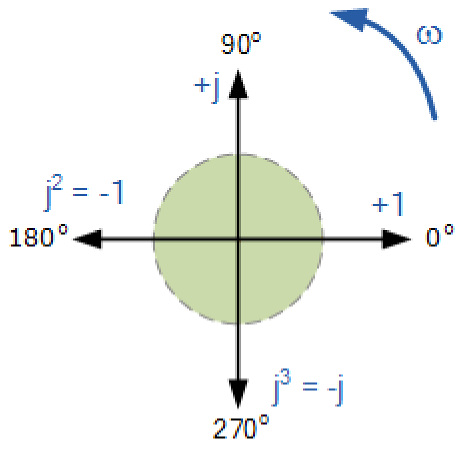
Vector Rotation

$$90^{\circ} \text{ rotation:} \quad j^1 = \sqrt{-1} = +j$$

$$180^{\circ} \text{ rotation:} \quad j^2 = \left(\sqrt{-1}\right)^2 = -1$$

$$270^{\circ} \text{ rotation:} \quad j^3 = \left(\sqrt{-1}\right)^3 = -j$$

$$360^{\circ} \text{ rotation:} \quad j^4 = \left(\sqrt{-1}\right)^4 = +1$$



So by multiplying an imaginary number by j^2 will rotate the vector by 180° anticlockwise, multiplying by j^3 rotates it 270° and by j^4 rotates it 360° or back to its original position. Multiplication by j^{10} or by j^{30} will cause the vector to rotate anticlockwise by the appropriate amount. In each successive rotation, the magnitude of the vector always remains the same.

Complex and Polar forms of Representation:

In Electrical Engineering there are different ways to represent a complex number either graphically or mathematically. One such way that uses the cosine and sine rule is called the Cartesian or Rectangular Form.

A complex number is represented by a real part and an imaginary part that takes the generalised form of:

$$Z=x+jy$$

Where

Z - is the Complex Number representing the Vector

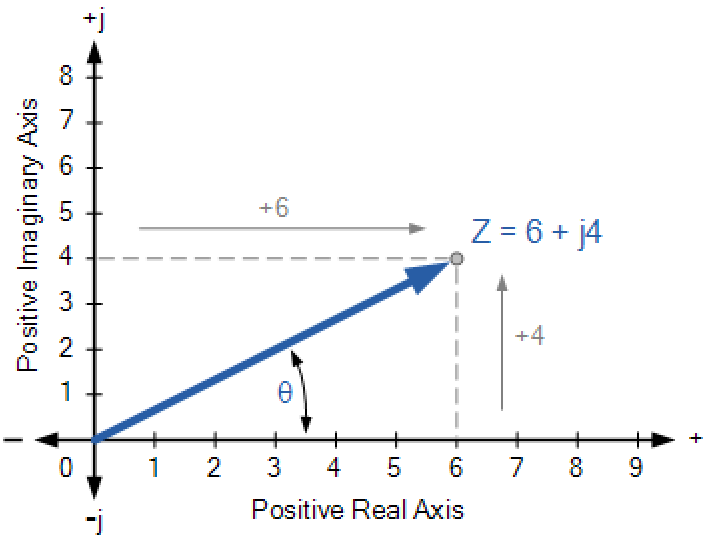
x - is the Real part or the Active component

y - is the Imaginary part or the Reactive component

j - is defined by $\sqrt{-1}$

In the rectangular form, a complex number can be represented as a point on a two-dimensional plane called the complex or s-plane. So for example, Z = 6 + j4 represents a single point whose coordinates represent 6 on the horizontal real axis and 4 on the vertical imaginary axis as shown.

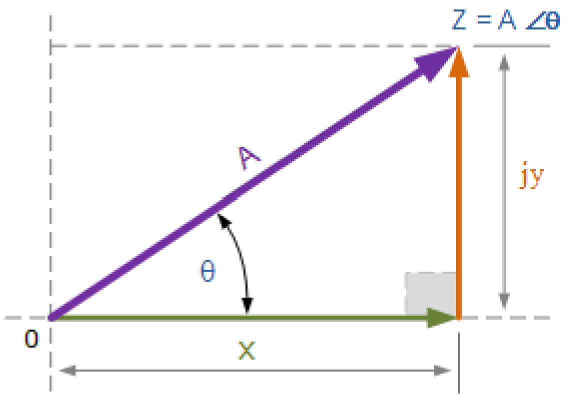
Complex Numbers using the Complex or s-plane:



Complex Numbers using Polar Form:

Unlike rectangular form which plots points in the complex plane, the Polar Form of a complex number is written in terms of its magnitude and angle. Thus, a polar form vector is presented as: Z = A ∠±θ, where: Z is the complex number in polar form, A is the magnitude or modulo of the vector and θ is its angle or argument of A which can be either positive or negative. The magnitude and angle of the point still remains the same as for the rectangular form above, this time in polar form the location of the point is represented in a "triangular form" as shown below.

Polar Form Representation of a Complex Number:



As the polar representation of a point is based around the triangular form, we can use simple geometry of the triangle and especially trigonometry and Pythagoras's Theorem on triangles to find both the magnitude and the angle of the complex number. As we remember from school, trigonometry deals with the relationship between the sides and the angles of triangles so we can describe the relationships between the sides as:

$$A^2=X^2+Y^2$$

$$A=\sqrt{X^2+Y^2}$$

Also $X=A\cos\theta$ $Y=A\sin\theta$

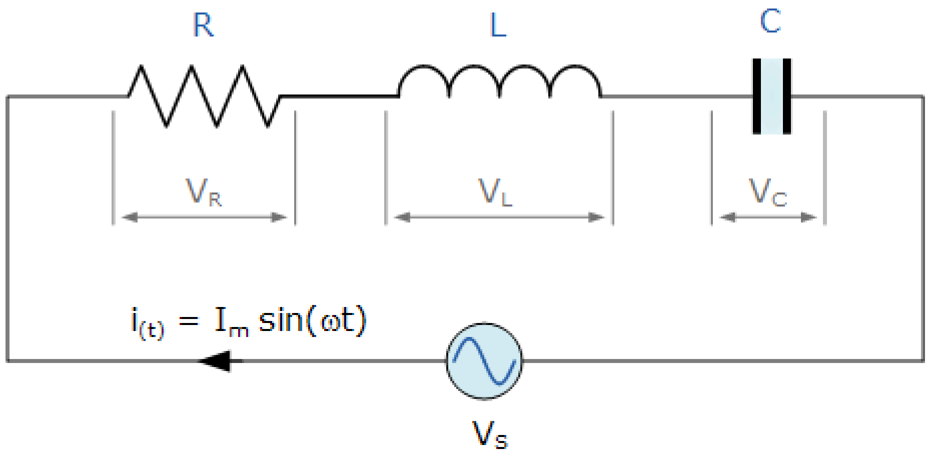
Using trigonometry again, the angle θ of A is given as follows.

$$\Theta=\tan^{-1}y/x$$

Then in Polar form the length of A and its angle represents the complex number instead of a point. Also in polar form, the conjugate of the complex number has the same magnitude or modulus it is the sign of the angle that changes, so for example the conjugate of $6\angle 30^o$ would be $6\angle -30^o$.

Steady state Analysis of Series RLC circuits:

Thus far we have seen that the three basic passive components: resistance (R), inductance (L), and capacitance (C) have very different phase relationships to each other when connected to a sinusoidal AC supply.



In a pure ohmic resistor the voltage waveforms are "in-phase" with the current. In a pure inductance the voltage waveform "leads" the current by 90^o, giving us the expression of: ELI. In a pure capacitance the voltage waveform "lags" the current by 90^o, giving us the expression of: ICE.

This phase difference,Φ depends upon the reactive value of the components being used and hopefully by now we know that reactance, (X) is zero if the circuit element is resistive, positive if the circuit element is inductive and negative if it is capacitive thus giving their resulting impedances as:

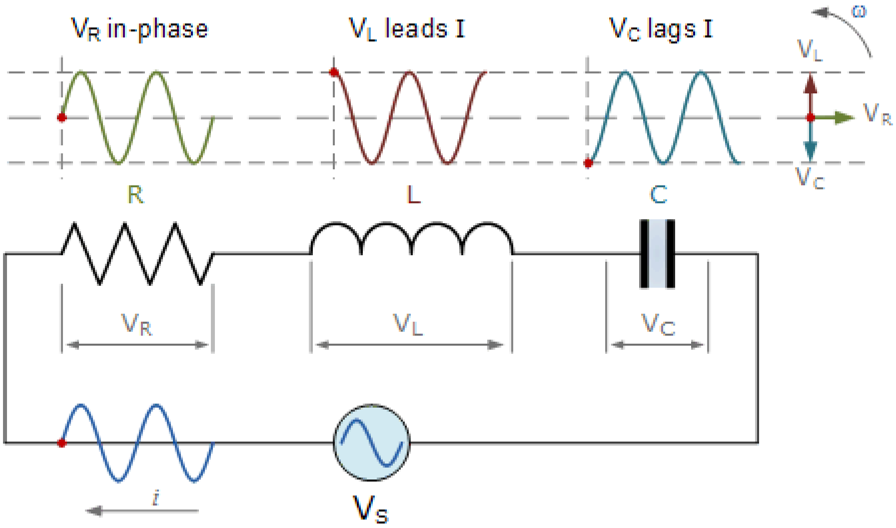
Element Impedance:

Circuit element	Resistsnce(R)	Reactance(X)	Impeadance(Z)
RESISTOR	R	0	Z_R=R RL0^0
INDUCTOR	L	WL	Z_L=WL L90^0
CAPACITOR	C	1/WC	Z_C=1/WC L-90^0

The series RLC circuit above has a single loop with the instantaneous current flowing through the loop being the same for each circuit element. Since the inductive and capacitive reactance's X_L and X_C are a function of the supply frequency, the sinusoidal response of a series RLC circuit will therefore vary with frequency, *ƒ*. Then the individual voltage drops across each circuit element of R, L and C element will be "out-of-phase" with each other as defined by:

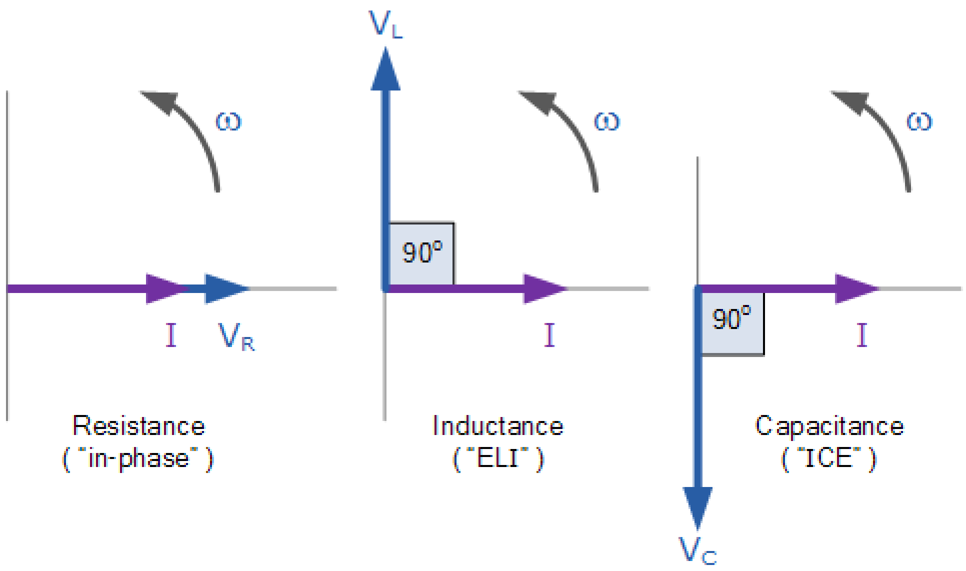
$i_{(t)} = I_{max} \sin(\omega t)$

The instantaneous voltage across a pure resistor, V_R is "in-phase" with current
The instantaneous voltage across a pure inductor, V_L "leads" the current by 90^o
The instantaneous voltage across a pure capacitor, V_C "lags" the current by 90^o
Therefore, V_L and V_C are 180^o "out-of-phase" and in opposition to each other



The amplitude of the source voltage across all three components in a series RLC circuit is made up of the three individual component voltages, V_R, V_L and V_C with the current common to all three components. The vector diagrams will therefore have the current vector as their reference with the three voltage vectors being plotted with respect to this reference as shown below.

Individual Voltage Vectors



This means then that we cannot simply add together V_R, V_L and V_C to find the supply voltage, V_S across all three components as all three voltage vectors point in different directions with regards to the current vector. Therefore we will have to find the supply voltage, V_S as the Phasor Sum of the three component voltages combined together vectorially.

Kirchoff's voltage law (KVL) for both loop and nodal circuits states that around any closed loop the sum of voltage drops around the loop equals the sum of the EMF's. Then applying this law to the these three voltages will give us the amplitude of the source voltage, V_S as.

Instantaneous Voltages for a Series RLC Circuit:

$$\text{KVL:} \quad V_S - V_R - V_L - V_C = 0$$

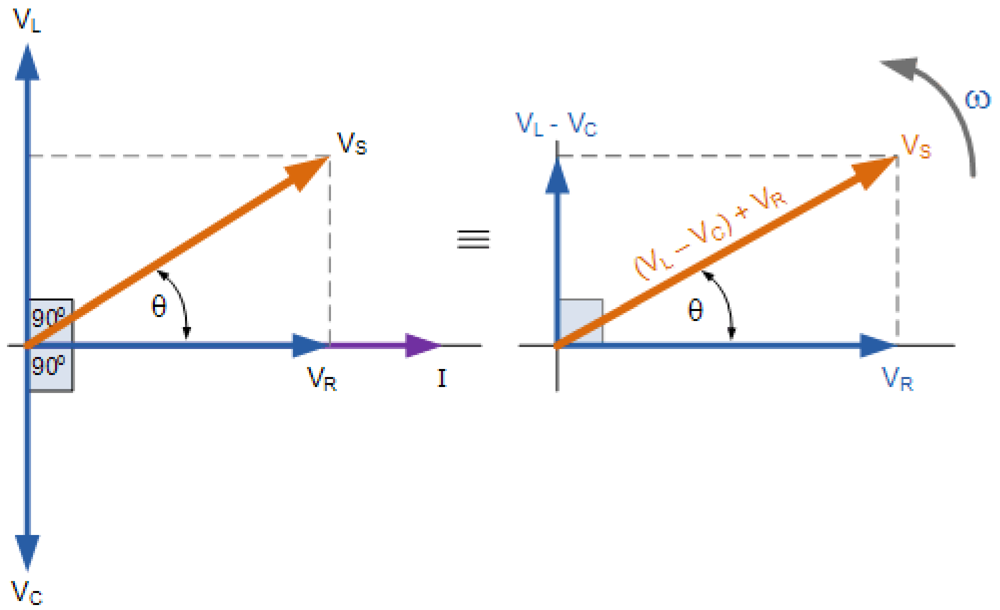
$$V_S - IR - L\frac{di}{dt} - \frac{Q}{C} = 0$$

$$\therefore V_S = IR + L\frac{di}{dt} + \frac{Q}{C}$$

The phasor diagram for a series RLC circuit is produced by combining together the three individual phasors above and adding these voltages vectorially. Since the current flowing through the circuit is common to all three circuit elements we can use this as the reference vector with the three voltage vectors drawn relative to this at their corresponding angles.

The resulting vector V_S is obtained by adding together two of the vectors, V_L and V_C and then adding this sum to the remaining vector V_R. The resulting angle obtained between V_S and i will be the circuits phase angle as shown below.

Phasor Diagram for a Series RLC Circuit:



We can see from the phasor diagram on the right hand side above that the voltage vectors produce a rectangular triangle, comprising of hypotenuse V_S, horizontal axis V_R and vertical axis $V_L – V_C$ Hopefully you will notice then, that this forms our old favourite the Voltage Triangle and we can therefore use Pythagoras's theorem on this voltage triangle to mathematically obtain the value of V_S as shown.

Voltage Triangle for a Series RLC Circuit:

$$V_S^2 = V_R^2 + (V_L - V_C)^2$$

$$V_S = \sqrt{V_R^2 + (V_L - V_C)^2}$$

Please note that when using the above equation, the final reactive voltage must always be positive in value, that is the smallest voltage must always be taken away from the largest voltage we cannot have a negative voltage added to V_R so it is correct to have $V_L – V_C$ or $V_C – V_L$. The smallest value from the largest otherwise the calculation of V_S will be incorrect.We know from above that the current has the same amplitude and phase in all the components of a series RLC circuit. Then the voltage across each component can also be described mathematically according to the current flowing through, and the voltage across each element as.

$$V_R = iR\sin(\omega t + 0^\circ) = i.R$$

$$V_L = iX_L\sin(\omega t + 90^\circ) = i.j\omega L$$

$$V_C = iX_C\sin(\omega t - 90^\circ) = i.\frac{1}{j\omega C}$$

By substituting these values into Pythagoras's equation above for the voltage triangle will give us:

$$V_R = I.R \quad V_L = I.X_L \quad V_C = I.X_C$$

$$V_S = \sqrt{(I.R)^2 + (I.X_L - I.X_C)^2}$$

$$V_S = I.\sqrt{R^2 + (X_L - X_C)^2}$$

$$\therefore V_S = I \times Z \quad \text{where: } Z = \sqrt{R^2 + (X_L - X_C)^2}$$

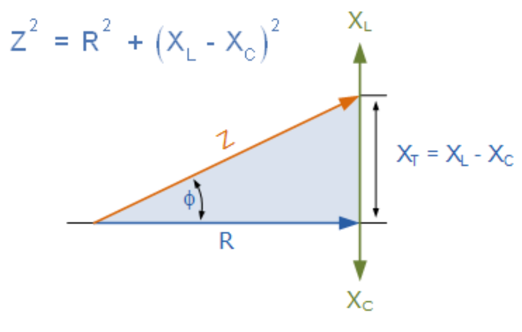
So we can see that the amplitude of the source voltage is proportional to the amplitude of the current flowing through the circuit. This proportionality constant is called the Impedance of the circuit which ultimately depends upon the resistance and the inductive and capacitive reactance's.

Then in the series RLC circuit above, it can be seen that the opposition to current flow is made up of three components, X_L, X_C and R with the reactance, X_T of any series RLC circuit being defined as: $X_T = X_L – X_C$ or $X_T = X_C – X_L$ with the total impedance of the circuit being thought of as the voltage source required to drive a current through it.

The Impedance of a Series RLC Circuit

As the three vector voltages are out-of-phase with each other, X_L, X_C and R must also be "out-of-phase" with each other with the relationship between R, X_L and X_C being the vector sum of these three components thereby giving us the circuits overall impedance, Z. These circuit impedance's can be drawn and represented by an Impedance Triangle as shown below.

The Impedance Triangle for a Series RLC Circuit



The impedance Z of a series RLC circuit depends upon the angular frequency, ω as do X_L and X_C If the capacitive reactance is greater than the inductive reactance, $X_C > X_L$ then the overall circuit reactance is capacitive giving a leading phase angle.

Likewise, if the inductive reactance is greater than the capacitive reactance, $X_L > X_C$ then the overall circuit reactance is inductive giving the series circuit a lagging phase angle. If the two reactance's are the same and $X_L = X_C$ then the angular frequency at which this occurs is called the resonant frequency and produces the effect of resonance

Then the magnitude of the current depends upon the frequency applied to the series RLC circuit. When impedance, Z is at its maximum, the current is a minimum and likewise, when Z is at its minimum, the current is at maximum. So the above equation for impedance can be re-written as:

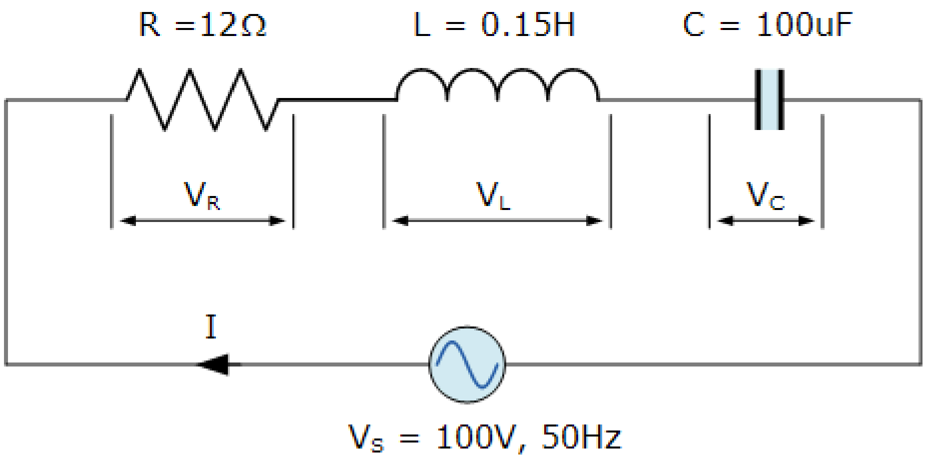
$$\text{Impedance, } Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

The phase angle, θ between the source voltage, V_S and the current, i is the same as for the angle between Z and R in the impedance triangle. This phase angle may be positive or negative in value depending on whether the source voltage leads or lags the circuit current and can be calculated mathematically from the ohmic values of the impedance triangle as:

$$\cos\phi = \frac{R}{Z} \qquad \sin\phi = \frac{X_L - X_C}{Z} \qquad \tan\phi = \frac{X_L - X_C}{R}$$

Series RLC Circuit Example

A series RLC circuit containing a resistance of 12Ω, an inductance of 0.15H and a capacitor of 100uF are connected in series across a 100V, 50Hz supply. Calculate the total circuit impedance, the circuits current, power factor and draw the voltage phasor diagram



Inductive Reactance, X_L.

$$X_L = 2\pi f L = 2\pi \times 50 \times 0.15 = 47.13\Omega$$

Capacitive Reactance, X_C.

$$X_C = \frac{1}{2\pi f C} = \frac{1}{2\pi \times 50 \times 100 \times 10^{-6}} = 31.83\Omega$$

Circuit Impedance, Z

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$Z = \sqrt{12^2 + (47.13 - 31.83)^2}$$

$$Z = \sqrt{144 + 234} = 19.4\Omega$$

Circuits Current, I.

$$I = \frac{V_S}{Z} = \frac{100}{19.4} = 5.14\text{Amps}$$

Voltages across the Series RLC Circuit, V_R, V_L, V_C.

$$V_R = I \times R = 5.14 \times 12 = 61.7\text{ volts}$$

$$V_L = I \times X_L = 5.14 \times 47.13 = 242.2\text{ volts}$$

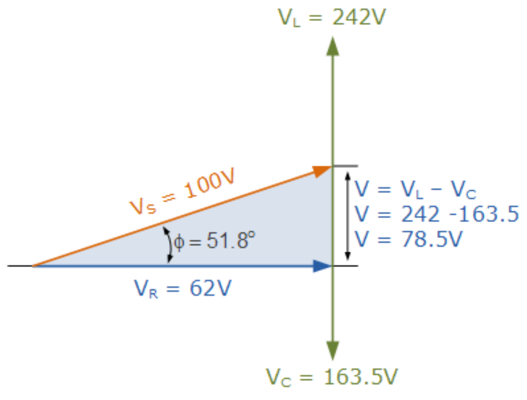
$$V_C = I \times X_C = 5.14 \times 31.8 = 163.5\text{ volts}$$

Circuits Power factor and Phase Angle, θ.

$$\cos\phi = \frac{R}{Z} = \frac{12}{19.4} = 0.619$$

$$\therefore \cos^{-1} 0.619 = 51.8^0 \text{ lagging}$$

Phasor Diagram.



Concept of Reactance, Impedance, Susceptance and Admittance:

Reactance is essentially inertia against the motion of electrons. It is present anywhere electric or magnetic fields are developed in proportion to applied voltage or current, respectively; but most notably in capacitors and inductors. When alternating current goes through a pure reactance, a voltage drop is produced that is 90^o out of phase with the current. Reactance is mathematically symbolized by the letter "X" and is measured in the unit of ohms (Ω).

Impedance is a comprehensive expression of any and all forms of opposition to electron flow, including both resistance and reactance. It is present in all circuits, and in all components. When alternating current goes through an impedance, a voltage drop is produced that is somewhere between 0^o and 90^o out of phase with the current. Impedance is mathematically symbolized by the letter "Z" and is measured in the unit of ohms (Ω), in complex form

Admittance is also a complex number as impedance which is having a real part, Conductance (G) and imaginary part, Susceptance (B).

$$Y = G + jB$$
$$Y \rightarrow Admittance\ in\ Siemens$$
$$G \rightarrow Conductance\ in\ Siemens = \frac{R}{R^2 + X^2}$$
$$B \rightarrow\ Susceptance\ in\ Siemens = -\frac{X}{R^2 + X^2}$$

(it is negative for capacitive susceptance and positive for inductive susceptance)

$$j^2 = -1$$
$$|\,Y\,| = \sqrt{G^2 + B^2} = \frac{1}{\sqrt{R^2 + X^2}}$$
$$\angle Y = arctan\left(\frac{B}{G}\right) = arctan\left(-\frac{X}{R}\right)$$

Susceptance (symbolized *B*) is an expression of the ease with which alternating current (AC) passes through a capacitance or inductance

Phase and phase difference:
Generally all sinusoidal waveforms will not pass exactly through the zero axis point at the same time, but may be "shifted" to the right or to the left of 0^o by some value when compared to another sine wave. Any sine wave that does not pass through zero at t = 0 has a phase shift.

The phase difference or phase shift as it is also called of a Sinusoidal Waveform is the angle Φ (Greek letter Phi), in degrees or radians that the waveform has shifted from a certain reference point along the horizontal zero axis. In other words phase shift is the lateral difference between two or more waveforms along a common axis and sinusoidal waveforms of the same frequency can have a phase difference.

The phase difference, Φ of an alternating waveform can vary from between 0 to its maximum time period, T of the waveform during one complete cycle and this can be anywhere along the horizontal axis between, $\Phi = 0$ to 2π (radians) or $\Phi = 0$ to 360^{o} depending upon the angular units used.

Phase difference can also be expressed as a *time shift* of τ in seconds representing a fraction of the time period, T for example, +10mS or – 50uS but generally it is more common to express phase difference as an angular measurement.

Then the equation for the instantaneous value of a sinusoidal voltage or current waveform we developed in the previous Sinusoidal Waveform will need to be modified to take account of the phase angle of the waveform and this new general expression becomes.

Phase Difference Equation

$$A_{(t)} = A_{max} \times \sin(\omega t \pm \Phi)$$

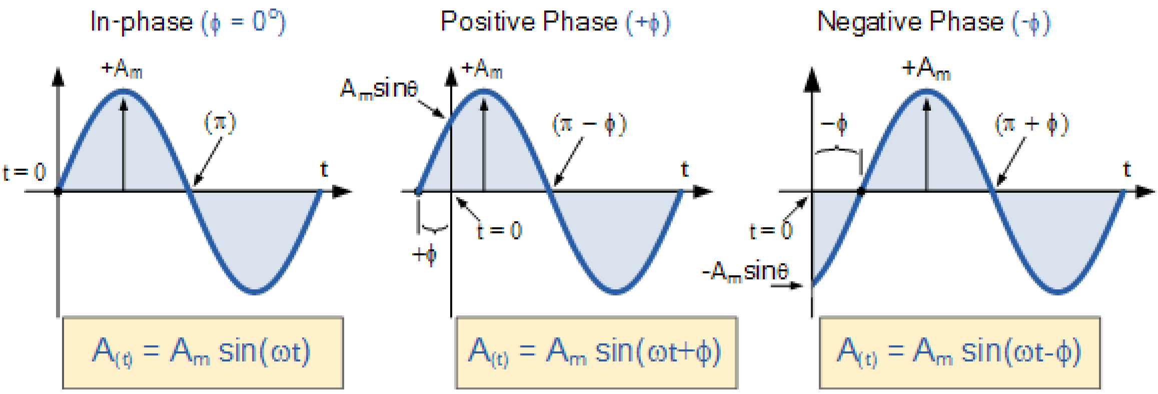
Where:

A_m - is the amplitude of the waveform.

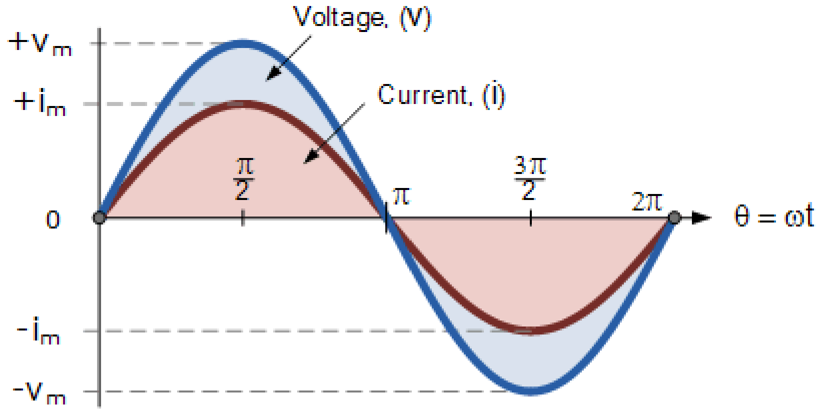
ωt - is the angular frequency of the waveform in radian/sec.

Φ (phi) - is the phase angle in degrees or radians that the waveform has shifted either left or right from the reference point

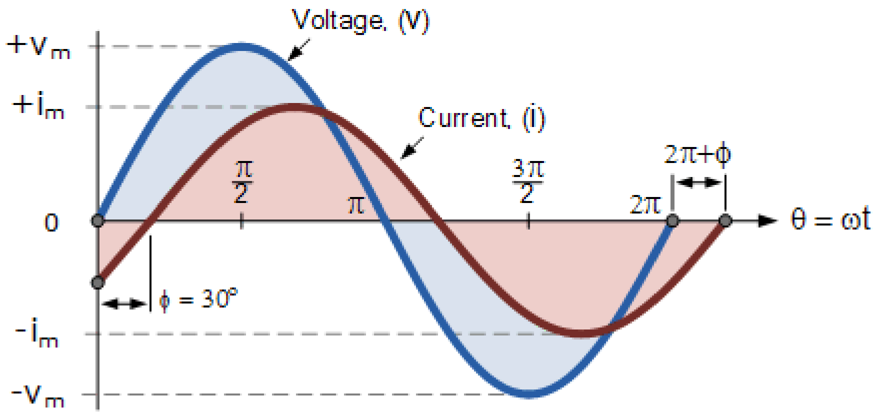
Phase Relationship of a Sinusoidal Waveform:



Two Sinusoidal Waveforms – “in-phase”



Phase Difference of a Sinusoidal Waveform:



The voltage waveform above starts at zero along the horizontal reference axis, but at that same instant of time the current waveform is still negative in value and does not cross this reference axis until 30^o later. Then there exists a Phase difference between the two waveforms as the current cross the horizontal reference axis reaching its maximum peak and zero values after the voltage waveform.

As the two waveforms are no longer “in-phase”, they must therefore be “out-of-phase” by an amount determined by phi, Φ and in our example this is 30^o. So we can say that the two waveforms are now 30^o out-of phase. The current waveform can also be said to be “lagging” behind the voltage waveform by the phase angle, Φ. Then in our example above the two waveforms have a Lagging Phase Difference so the expression for both the voltage and current above will be given as.

$$\text{Voltage}, (v_t) = V_m \sin\omega t$$

$$\text{Current}, (i_t) = I_m \sin(\omega t - \theta)$$

where, i lags v by angle Φ

Likewise, if the current, i has a positive value and crosses the reference axis reaching its maximum peak and zero values at some time before the voltage, v then the current waveform will be "leading" the voltage by some phase angle. Then the two waveforms are said to have a Leading Phase Difference and the expression for both the voltage and the current will be.

$$\text{Voltage}, (v_t) = V_m \sin\omega t$$

$$\text{Current}, (i_t) = I_m \sin(\omega t + \theta)$$

where, i leads v by angle Φ

Concept of power factor, real, reactive and complex power:

Complex Power is defined as the product of Voltage phasor and conjugate of current phasor

If S is the complex power then,

$$S = V \,.\, I^*$$

V is the phasor representation of voltage and I* is the conjugate of current phasor.

So if V is the reference phasor then V can be written as $|V| \angle 0$.
(Usually one phasor is taken reference which makes zero degrees with real axis. It eliminates the necessity of introducing a non zero phase angle for voltage)
Let current lags voltage by an angle φ, so $I = |I| \angle{-\varphi}$
(current phasor makes -φ degrees with real axis)

$$I^* = |I| \angle\varphi$$

So,

$$S = |V| \; |I| \angle(0+\varphi) = |V| \; |I| \angle\varphi$$

(For multiplication of phasors we have considered polar form to facilitate calculation)

Writing the above formula for S in rectangular form we get

$$S = |V| \; |I| \cos\varphi \; + \; j \; |V| \; |I| \sin\varphi$$

The real part of complex power S is $|V| \, |I| \cos\varphi$ which is the real power or average power and the imaginary part $|V| \, |I| \sin\varphi$ is the reactive power.

So, $\quad S = P + j\,Q$

Where $\quad P = |V|\,|I| \cos\varphi \quad$ and $\quad Q = |V|\,|I| \sin\varphi$

P is measured in watt and Q is measured in VoltAmp-Reactive or VAR. In power systems instead of these smaller units larger units like Megawatt, MVAR and MVA is used.

The ratio of real power and apparent power is the power factor

$$\text{power factor} = \cos \varphi = |P| / |S|$$

$$= |P| / \sqrt{(P^2+Q^2)}$$