Discrete Fourier Transform (DFT):

The discrete Fourier transform of a finite-length sequence x(n) is defined as

$$X(k) = \sum_{k=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad 0 \le k \le N-1$$

X(k) is periodic with period N i.e., X(k+N) = X(k).

Inverse Discrete Fourier Transform (IDFT):

The inverse discrete Fourier transform of X(k) is defined as

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \quad 0 \le n \le N-1$$

For notation purpose discrete Fourier transform and inverse Fourier transform can be represented by

$$X(k) = DFT\,[x(n)]$$
$$x(n) = IDFT\,[X(k)]$$

Formula:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \frac{kn}{N}}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi \frac{kn}{N}}$$

Where K and n are in the range of 0 ,1,2……N-1

For example, if N=4

K= 0,1,2,3
N=0,1,2,3

Alternative Formula:

$$X(k) = \sum_{n=0}^{N-1} x(n)W^{kn} \quad \longleftarrow W = e^{-j\frac{2\pi}{N}}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W^{-kn}.$$

3.1 Properties of DFT:

Periodicity property:

If X(k) is the N-point DFT of x(n), then

$X(k+N)=X(k)$

Linearity property:

If $X_1(k)=DFT[x_1(n)]$ & $X_2(k)=DFT[x_2(n)]$, then

$DFT[a_1x_1(n)+a_2x_2(n)]=a_1X_1(k)+a_2X_2(k)$

Convolution property:

If $X_1(k) = DFT[x_1(n)]$ & $X_2(k) = DFT[x_2(n)]$, then

$DFT[x_1(n) \textcircled{N} x_2(n)] = X_1(k)X_2(k)$

Where $\textcircled{N}$ indicates N-point circular convolution.

Multiplication property:

If $X_1(k) = DFT[x_1(n)]$ & $X_2(k) = DFT[x_2(n)]$, then

$DFT[x_1(n)x_2(n)] = (1/N)[X_1(k) \textcircled{N} X_2(k)]$

Where $\textcircled{N}$ Indicates N-point circular convolution.

Time reversal property:

If X(k) is the N-point DFT of x(n), then

$DFT[x(N-n)] = X(N-k)$

Time shift property:

If X(k) is the N-point DFT of x(n), then

$$DFT[x(n-m)] = e^{-j2\pi mk/N}X(k)$$

Symmetry properties:

If $x(n)=x_R(n)+jx_I(n)$ is N-point complex sequence and $X(k)=X_R(k)+jX_I(k)$ is the N-point DFT of x(n) where $x_R(n)$ & $x_I(n)$ are the real & imaginary parts of x(n) and $X_R(k)$ & $X_I(k)$ are the those of X(k), then

(i) $DFT[x^*(n)]=X^*(N-k)$
(ii) $DFT[x^*(N-n)]=X^*(k)$
(iii) $DFT[x_R(n)]=(1/2)[X(k)+X^*(N-k)]$
(iv) $DFT[x_I(n)]=(1/2j)[X(k)-X^*(N-k)]$
(v) $DFT[x_{ce}(n)]=X_R(k)$ where $x_{ce}(n)=(1/2)[x(n)+x^*(N-n)]$
(vi) $DFT[x_{co}(n)]=jX_I(k)$ where $x_{co}(n)=(1/2)[x(n)-x^*(N-n)]$

If x(n) is real, then

(i) If x(n) is real, then
   a. $X(k)=X^*(N-k)$
   b. $X_R(k)=X_R(N-k)$
(ii) If x(n) is real, then
   a) $X(k)=X^*(N-k)$
   b) $X_R(k)=X_R(N-k)$
   c) $X_I(k)= -X_I(N-k)$
   d) $|X(k)|=|X(N-k)|$
   e) $|X(k)|=|X(N-k)|$
   f) $\angle X(k)= -\angle X(N-k)$
(i) $DFT[x_{ce}(n)]=X_R(k)$ where $x_{ce}(n)=(1/2)[x(n)+x(N-n)]$
(ii) $DFT[x_{co}(n)]=jX_I(k)$ where $x_{co}(n)=(1/2)[x(n)-x(N-n)]$

Problem 1:

Find the DFT of a sequence x(n)= {1,1,0,0} and find the IDFT of Y(K)= {1,0,1,0}

Let us assume $N = L = 4$.

We have $X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} \quad k = 0, 1, \ldots, N-1$

$$X(0) = \sum_{n=0}^{3} x(n) = x(0) + x(1) + x(2) + x(3)$$

$$= 1 + 1 + 0 + 0 = 2$$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j\pi n/2} = x(0) + x(1)e^{-j\pi/2} + x(2)e^{-j\pi} + x(3)e^{-j3\pi/2}$$

$$= 1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}$$

$$= 1 - j$$

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0) + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$= 1 + \cos\pi - j\sin\pi$$

$$= 1 - 1 = 0$$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j3n\pi/2} = x(0) + x(1)e^{-j3\pi/2} + x(2)e^{-j3\pi} + x(3)e^{-j9\pi/2}$$

$$= 1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$$

$$= 1 + j$$

$$X(k) = \{2, 1-j, 0, 1+j\}$$

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)e^{j2\pi nk/N} \quad n = 0, 1, \dots N-1$$

$$y(0) = \frac{1}{4}\sum_{k=0}^{3} Y(k) \quad n = 0, 1, 2, 3$$

$$= \frac{1}{4}\left[y(0) + y(1) + y(2) + y(3)\right]$$

$$= \frac{1}{4}[1 + 0 + 1 + 0]$$

$$= 0.5$$

$$y(1) = \frac{1}{N} \sum_{k=0}^{3} Y(k) e^{j\pi k/2}$$

$$y(1) = \frac{1}{4} \left[Y(0) + Y(1) e^{j\pi/2} + Y(2) e^{j\pi} + Y(3) e^{j3\pi/2} \right]$$

$$= \frac{1}{4} [1 + 0 + \cos\pi + j\sin\pi + 0]$$

$$= \frac{1}{4} [1 + 0 - 1 + 0] = 0$$

$$y(2) = \frac{1}{4} \left[Y(0) + Y(1) e^{j\pi} + Y(2) e^{j2\pi} + Y(3) e^{j3\pi} \right]$$

$$= \frac{1}{4} [1 + 0 + \cos 2\pi + j\sin 2\pi + 0]$$

$$= \frac{1}{4} [1 + 0 + 1 + 0] = 0.5$$

$$y(3) = \frac{1}{4} \left[Y(0) + Y(1) e^{j3\pi/2} + Y(2) e^{j3\pi} + Y(3) e^{j9\pi/2} \right]$$

$$= \frac{1}{4} [1 + 0 + \cos 3\pi + j\sin 3\pi + 0]$$

$$= \frac{1}{4} [1 + 0 + (-1) + 0] = 0$$

$$y(n) = \{0.5, 0, 0.5, 0\}$$