Z-Transform

Analysis of continuous time LTI systems can be done using z-transforms. It is a powerful mathematical tool to convert differential equations into algebraic equations.

The bilateral (two sided) z-transform of a discrete time signal x(n) is given as

$$Z.T[x(n)] = X(Z) = \Sigma_{n=-\infty}^{\infty} x(n) z^{-n}$$

The unilateral (one sided) z-transform of a discrete time signal x(n) is given as

$$Z.T[x(n)] = X(Z) = \Sigma_{n=0}^{\infty} x(n) z^{-n}$$

Z-transform may exist for some signals for which Discrete Time Fourier Transform (DTFT) does not exist.

Concept of Z-Transform and Inverse Z-Transform

Z-transform of a discrete time signal x(n) can be represented with X(Z), and it is defined as

$$X(Z) = \Sigma_{n=-\infty}^{\infty} x(n) z^{-n} \ldots\ldots (1)$$

If $Z = re^{j\omega}$ then equation 1 becomes

$$X(re^{j\omega}) = \Sigma_{n=-\infty}^{\infty} x(n)[re^{j\omega}]^{-n}$$

$$= \Sigma_{n=-\infty}^{\infty} x(n)[r^{-n}]e^{-j\omega n}$$

$$X(re^{j\omega}) = X(Z) = F.T[x(n)r^{-n}] \ldots\ldots (2)$$

The above equation represents the relation between Fourier transform and Z-transform

$$X(Z)|_{z=e^{j\omega}} = F.T[x(n)].$$

Inverse Z-transform:

$$X(re^{j\omega}) = F.T[x(n)r^{-n}]$$

$$x(n)r^{-n} = F.T^{-1}[X(re^{j\omega}]$$

$$x(n) = r^n F.T^{-1}[X(re^{j\omega})]$$

$$= r^n \frac{1}{2\pi} \int X(re^j\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int X(re^j\omega)[re^{j\omega}]^n d\omega \ldots\ldots (3)$$

Substitute $re^{j\omega} = z$.

$$dz = jre^{j\omega} d\omega = jzd\omega$$

$$d\omega = \frac{1}{j} z^{-1} dz$$

Substitute in equation 3.

$$3 \rightarrow x(n) = \frac{1}{2\pi} \int X(z) z^n \frac{1}{j} z^{-1} dz = \frac{1}{2\pi j} \int X(z) z^{n-1} dz$$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$x(n) = \frac{1}{2\pi j} \int X(z) z^{n-1} dz$$

Initial Value and Final Value Theorems

Initial value and final value theorems of z-transform are defined for causal signal.

Initial Value Theorem

For a causal signal x(n), the initial value theorem states that

$$x(0) = \lim_{z\to\infty} X(z)$$

This is used to find the initial value of the signal without taking inverse z-transform

Final Value Theorem

For a causal signal x(n), the final value theorem states that

$$x(\infty) = \lim_{z\to 1} [z-1] X(z)$$

This is used to find the final value of the signal without taking inverse z-transform

Region of Convergence (ROC) of Z-Transform

The range of variation of z for which z-transform converges is called region of convergence of z-transform.

Properties of ROC of Z-Transforms

- ROC of z-transform is indicated with circle in z-plane.
- ROC does not contain any poles.
- If x(n) is a finite duration causal sequence or right sided sequence, then the ROC is entire z-plane except at $z = 0$.
- If x(n) is a finite duration anti-causal sequence or left sided sequence, then the ROC is entire z-plane except at $z = \infty$.
- If x(n) is a infinite duration causal sequence, ROC is exterior of the circle with radius a. i.e. $|z| > a$.
- If x(n) is a infinite duration anti-causal sequence, ROC is interior of the circle with radius a. i.e. $|z| < a$.
- If x(n) is a finite duration two sided sequence, then the ROC is entire z-plane except atz $= 0$ & $z = \infty$.

Discrete Time Fourier Transforms (DTFT)

Here we take the exponential signals to be $\{e^{j\omega n}\}$ where is a real number. The representation is motivated by the Harmonic analysis, but instead of following the historical development of the representation we give directly the defining equation. Let be discrete time signal $\{x[n]\}$ such that $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ that is sequence is absolutely summable $\{x[n]\}$.

The sequence $\{x[n]\}$ can be represented by a Fourier integral of the form.

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \qquad \text{..................(1)}$$

Where,

$$x(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \qquad \text{..................(2)}$$

Equation (1) and (2) give the Fourier representation of the signal. Equation (1) is referred as synthesis equation or the inverse discrete time Fourier transform (IDTFT) and equation (2) is Fourier transform in the analysis equation. Fourier transform of a signal in general is a complex valued function, we can write

$$X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})$$

where $|X(e^{j\omega})|$ is magnitude and $\angle X(e^{j\omega})$ is the phase of. We also use the term Fourier spectrum or simply, the spectrum toreferto. Thus, $|X(e^{j\omega})|$ is called the magnitude spectrum and $\angle X(e^{j\omega})$ is called the phase spectrum. From equation (2) we can see that $X(e^{j\omega})$ is a periodic function with period i . e . 2π. We can interpret (1) as Fourier coefficients in the representation of a periodic function. In the Fourier series analysis, our attention is on the periodic function, here we are concerned with the representation of the signal. So, the roles of the two equations are interchanged compared to the Fourier series analysis of periodic signals.

Now we show that if we put equation (2) in equation (1) we indeed get the signal.
Let

$$\hat{x}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m} \right) e^{+j\omega n} d\omega$$

where we have substituted $X(e^{j\omega})$ from (2) into equation (1) and called the result as. Since we have used n as index on the left-hand side, we have used m as the index variable forthe sum defining the Fourier transform. Under our assumption that $\{x[n]\}$ sequence is absolutely summable we can interchange the order of integration and summation. Thus

$$\hat{x}[n] = \sum_{m=-\infty}^{\infty} x[m] \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{+j\omega(n-m)} d\omega \right)$$

Example: Let

$$\{x[n]\} = \{a^n u[n]\}$$

Fourier transform of this sequence will exist if it is absolutely summable. We have

$$\sum_{n=-\infty}^{\infty} |x[n]| = \sum_{n=0}^{\infty} |a|^n$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} \left(ae^{-j\omega}\right)^{-n} = \frac{1}{1-ae^{-j\omega}}.$$

The magnitude and phase for this example are show in the figure below, where $a > 0$ and $a < 0$ are shown in (a) and (b).