UNIT V NON LINEAR PROGRAMMING PROBLEM

Unconstrained external problems - Equality constraints - Lagrangean method - Kuhn – Tucker conditions – Simple problems - Jacobian methods

I. UNCONSTRAINED EXTERNAL PROBLEMS

Maxima and minima for a function of one variable

[Eg: Newton Raphson method]

Maxima and minima for a function of two variables

Constrained External problems

Eg: Lagrangean method

External problem with equality constraints

Eg: Lagrangean method

Constrained external problem with more than one equality constraint

Eg: Bordered Hessian Matrix

Constraint external problem with inequality constriants Eg: **Kuhn Tucker conditions**

Newton Raphson method:

Newton Raphson formula is given by

$X_{k+1} = xk-\frac{f(x_k)}{f'(x_k)}$, k =0,1,2….

Unconstrained External Problem:

Example:

Find the stationary points of f(x)= $4x^4-x^2+5$ and determine the nature of the stationary points.

Solution:

$f(x)= 4x^4-x^2+5$ --------------(1)

$f'(x)= 16x^3-2x$ --------------(2)

Stationary points are given by $f'(x) = 0$

$16x^3-2x = 0$

$2x(8x^2-1) =0$

$x=0$ or $8x^2-1 =0$

X=0 or x=±122There are three stationary values

$x=0$, $x= +\frac{1}{2\sqrt{2}}$, $x= -\frac{1}{2\sqrt{2}}$

To determine the nature of these values

$f''(x)= 48x^2-2$ -------(3)

Case(i) :

Consider x=0

$f''(0)=-2$ from (3)

Case(ii):

Consider x=$\frac{1}{2\sqrt{2}}$ from (3) ---⟶ f′′($\frac{1}{2\sqrt{2}}$) $= 48X(\frac{1}{8})$ -2 =4>0

x=$\frac{1}{2\sqrt{2}}$ minimises f(x) and the minimum value of f(x) is

f($\frac{1}{2\sqrt{2}}$) = 4 X ($\frac{1}{64}$) –($\frac{1}{8}$) +5 = $\frac{79}{16}$

Case(iii):

Consider x=$-\frac{1}{2\sqrt{2}}$

From (3) ------⟶ f′′($-\frac{1}{2\sqrt{2}}$) =48X($\frac{1}{2}$) -2 ; 4>0

x=$-\frac{1}{2\sqrt{2}}$ also minimizes f(x) and the minimum value f(x) = $\frac{79}{16}$

NOTE:

Stationary values are **x=0, x= +** $\frac{1}{2\sqrt{2}}$**, x= -** $\frac{1}{2\sqrt{2}}$

Example:

Determine the maximum and minimum value of the function f(x) = (3x-4)² (2x-3)²

Solution:

f(x) = $(3x-4)^2 (2x-3)^2$ ---------(1)

f′(x)=u′v+uv′

f′(x)=$2(3x-4)^3(2x-3)^2$+(3x-4)2X2(2x-3).2

=6(3x-4) (2x-3)2+4(3x-4)2 (2x-3)

=(3x-4) (2x-3)[6(2x-3)+4(3x-4)

=2(3x-4) (2x-3)[6x-9+6x-8]

=2(3x-4) (2x-3)(12x-17) ------------(2)

Stationary points/ Extreme points

f′(x)=0

2(3x-4) (2x-3)(12x-17)=0

x=$\frac{4}{3}$; x = x=$\frac{3}{2}$; x=$\frac{17}{12}$

Nature of stationary points

f′′(x)=3(2x-3)(24x-34)+2(3x-4)(24x-34)+24(3x-4)(2x-3) --------(3)

case(i):

x=$\frac{4}{3}$; from(3)

f′′(x)=3(2. $\frac{4}{3}$ -3) $(24.\frac{4}{3} - 34) + 0 + 0$

=3($\frac{8-9}{3}$)(-2)

=(-1)(-2) = 2>0

x=$\frac{4}{3}$ minimizes f(x)

from (1) f($\frac{4}{3}$) $= (3.\frac{4}{3} - 4)2\ (2.\frac{4}{3} - 3)^2$ =0

case (ii): x=$\frac{3}{2}$

from (3) f′′($\frac{3}{2}$) = 0+$(3.\frac{3}{2} - 4)(24.\frac{3}{2} - 32) + 0$

$=2(\frac{9-8}{2})(2)$; 2>0

Hence, $x=\frac{3}{2}$ minimizes f(x)

Case(iii): $x=\frac{17}{12}$

From (3) $f''(\frac{17}{12}) = 0 + 0 + 24(3.\frac{17}{12} - 4)(2.\frac{17}{12} - 3)$

$=24(\frac{51-48}{12})(\frac{17-18}{6})$

$=24(\frac{3}{12})(-\frac{1}{6})$

$=-1<0$

Hence $x=\frac{17}{12}$ maximizes f(x)

From (1) $f(\frac{17}{12}) = (3.\frac{17}{12} - 4)^2(2.\frac{17}{12} - 3)^2$

$= (\frac{17-16}{4})^2 (\frac{17-18}{6})^2$

$=\frac{1}{576}$

$x=\frac{17}{12}$; $f(x)= \frac{1}{576}$

Stationary Points/Extreme Points	Status	f(x) Maxima /Minima
$\frac{4}{3}$	>0 Minimizes	0 Minima
$\frac{3}{2}$	>0 Minimizes	0 Minima
$\frac{17}{12}$	< Maximizes	$\frac{1}{576}$ Maxima

NEWTON RAPHSON METHOD

The necessary condition for y=f(x) to have an extremum is f′(x)=0. Solving this equation may be very difficult and we should be satisfied with a reasonably approximate value of the roots of the equation f′(x)=0. There are many numerical methods for solving f′(x)= 0. One standard method studied in earlier semesters in numerical method is Newton Raphson method.

If x_0 is an initial approximation of a root of f′(x)=0 chosen properly in the vicinity of the root α of f′(x)=0, a≤ α ≤b. so as to ensure the convergence of the approximations then the Newton Raphson formula is given by

$X_{k+1} = xk-\frac{f(x_k)}{f'(x_k)}$, k =0,1,2….

When the successive iterations x_k and x_k+1 are approximately equal within a specified degree of accuracy then the convergence occur. Newton Raphson may be used to determine the extreme value.

Example:
By using the Newton-Raphson's method find the positive root of the quadratic equation 5x2 + 11x – 17 = 0 correct to 3 significant figures.
Solution:

$f(x)= 5x^2+11x-17$

$f'(x)=10x+11$

$f(x1)=f(1)=5*(1)2+11*(1)-17=5+11-17=-1$

$f'(x1)=f'(1)=10*(1)+11=21$

$x2=x1-\frac{f(x1)}{f'*(x1)}=1-\frac{f(1)}{f'(1)}=1-(-\frac{1}{21})=1+\frac{1}{21}=1.0476$

$x2= 1.0476$

$f(x2)=f(1.0476)=5*(1.0476)2+11*(1.0476)-17=0.1133$

$f'(x2)=f'(1.0476)=10*(1.0476)+11=10.476+11=21.476$

Checking it by the quadratic formula:

$x=-11\pm\frac{\sqrt{112-44*5*(-17)}}{2*5}$

$x=\frac{-11\pm}{10}\ \sqrt{461}$

Use the + to get the positive root:

$x=1.0476$

So we only need one iteration of the Newton-Raphson method to get it to three significant figures, for what we had then would have rounded to 1.05.

Example:
Investigate f(x)= x⁴-2x²-16x+1 for maxima and minima use Newton-Raphson method to determine the extreme value to 3 decimal places.
Solution:

$f(x)= x^4-2x^2-16x+1$ ---------(1)

$f'(x)=4x^3-4x-16$

$f'(x)=0$ gives

$4x^3-4x-16=0$

New $f(x)= x^3-x-4=0$ ----------(2)

Using Descartes rule of signs there is at most one positive root and no negative root. The other roots are complex. The positive roots lies between 1 and 2 since,

$f'(1)<o$ and $f'(2)>0$

This is a real root between 1 and 2. Take the initial approximation as 'x_0'

$x_0 = \frac{1+2}{2} = 1.5$ and use the Newton Raphson formula,

$f(x)= x^3-x-4=0$; k =0,1,2….

Newton Raphson formula is given by,

$X_{k+1} = xk-\frac{f(x_k)}{f'(x_k)}$, k =0,1,2….

x_k	$X_{k+1} = \frac{(2x_{k+}^3 x_{k+4})}{(3x_{k-1}^2)}$
x_0	1.5
x_1	1.87
x_2	1.80
x_3	1.796
x_4	1.796

The only extreme value is 1.796 correct to is 3 decimal places. Since two consecutive iterations agree at k=3,4 $f''(x)= 12x^2-4$

Hence, $f''(1.796) = 12(1.796)^2 -4>0$

x= 1.796 minimize f(x)

$f(x)=x^4-2x^2-16x+1$

Min $f(x) = (1.796)^4-2(1.796)^2-16(1.796)+1$

$f(x)=-23.783$

MAX AND MIN FOR AFUNCTION OF TWO VARIABLES:

NECESSARY CONDITIONS:

$r= \frac{\partial^2 z}{\partial x^2}$ $s=\frac{\partial^2 z}{\partial x \partial y}$ $t=\frac{\partial^2 z}{\partial y^2}$

(i) f attains a maximum at an extreme point (a,b) if $rt-s^2>0$ and $r<0$ at that point.

(ii) f attains a minimum of (a,b) if $rt-s^2>0$ and $r>0$ at that point.

(iii) f has a saddle point at (a,b) if $rt-s^2<0$.

(iv) If $rt-s^2=0$ further investigation is required to determine the nature of the extreme point.

STEPS FOR FINDING THE EXTREME OF Z= F(X,Y):

Step 1: Find $\frac{\partial z}{\partial x}$and $\frac{\partial z}{\partial y}$

Step 2: Solve $\frac{\partial z}{\partial y}$=0 and $\frac{\partial z}{\partial y}$ =0

The solution gives the at the critical points or stationary points of z=f(x,y)

Step 3: Calculate r,s and t at the critical point.

Step 4:

(i) if $rt-s^2>0$ and $r<0$ then f has a maximum at the critical point.

(ii) if $rt-s^2>0$ and $r<0$ then f has a minimum at the critical point.

(iii) if $rt-s^2<0$ f has neither a maximum nor minimum. It has a saddle point.

(iv) if $rt-s^2=0$ further investigation is required.

Example:

Investigate for maxima, minima and saddle point for the function $z=x^4+y^4-y^2-x^2+1$

Solution:

$Z= x^4+y^4-y^2-x^2+1$ ----------------------(1)

$\frac{\partial z}{\partial x}=4x^3-2x$

$\frac{\partial z}{\partial y}= 4y^3-2y$

$\frac{\partial z}{\partial x}=0$ gives, $4x^3-2x=0$

i.e $2x(2x^2-1)=0$

$x=0$ or $x= \pm\frac{1}{\sqrt{2}}$

$\frac{\partial z}{\partial y}=0$; $4y^3-2y=0$

$y=0$ or $y=\pm\frac{1}{\sqrt{2}}$

$r= \frac{\partial^2 z}{\partial x^2}= 12x2-2$ ----------------(2)

$t= \frac{\partial^2 z}{\partial y^2}=12y2-2$ ----------------(3)

$s=\frac{\partial^2 z}{\partial x\partial y} =0$

$rt-s^2 = (12x^2-2)(12y^2-2)-0$

$rt-s^2 =(12x^2-2)(12y^2-2)$

$rt-s^2=4(6x^2-1)(6y^2-1)$ -------------(4)

The extreme or critical points are (0,0) , $(0, +\frac{1}{\sqrt{2}})$,$(0,-\frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}},0)$,$(-\frac{1}{\sqrt{2}},0)$, $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$,

$(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, $(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$

Case i:

At(0,0) $rt-s^2 =4>0$

$r=-2<0$ from (2)

Z is maximum at (0,0) and max z=1

Case ii:

At$(0,\frac{1}{\sqrt{2}})$

$rt-s^2 = -4 X 2 = -8<0$

Hence, $(0,\frac{1}{\sqrt{2}})$ **is a saddle point.**

Case iii:

At$(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}})$

$rt-s^2 = 4X3X2 = 16>0$

$r=2>0$

Z is minimum at $(\pm\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}})$

Minimum value of z=1.

EQUALITY CONSTRAINTS

CONSTRAINED EXTERNAL PROBLEMS

Eg: Lagrangean method:

The most common method of solving external problems having continuous differentiable objective function as well as constraint functions with respect to the decision variables is the Lagrangean multiplier method.

Lagrangean multiplier method can be illustrated by the following simple two variable problems with one constraint.

Maximize or minimize Z= $f(x_1,x_2)$

Subject to $g(x_1,x_2) \leq b$

$x_1, x_2 \geq 0$

Step 1:

The constraint is replaced another function $h(x_1,x_2)$such that $h(x_1,x_2)=g(x_1,x_2) - b=0$

The problem now becomes

Maximize or minimize $Z=f(x_1,x_2)$

Subject to $h(x_1,x_2)=0$

$x_1,x_2 \geq 0$

Step 2:

The Lagrangean function L can be constructed as

$L(x_1,x_2,\lambda) = f(x_1,x_2) - \lambda h(x_1,x_2)$

Where λ is called the Lagrangean multiplier a constant.

For determining whether the solution results in maximization or minimization of the objective function find the first n-1 principal minors of the following determinant

$$\Delta_{n+1}=\begin{bmatrix} 0 & \frac{\partial h}{\partial x_1} & & \frac{\partial h}{\partial x_2}\,\frac{\partial h}{\partial x_n} & \\ \frac{\partial h}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2}-\lambda\frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 f}{\partial x1\partial x2}-\lambda\frac{\partial^2 h}{\partial x1\partial x2} & & x1\partial x2 \vdots \\ & \ldots & & & \end{bmatrix}$$

LAGRANGEAN METHOD

Constrained external problem with one equality constraint:

Example problem:

Use the Lagrangian method to maximize the function

$f(x; y) = xy$

subject to the constraint

$x + 2y \leq 200$

Solution

$L=xy- \lambda(x+2y-200)$

$\frac{\partial L}{\partial x}$ = y- λ(1) = 0 → (1)

$\frac{\partial L}{\partial y}$ = x- λ(2) = 0 → (2)

$\frac{\partial L}{\partial \lambda}$ = -(x+2y-200) = 0

x+2y = 200 → (3)

λ = y

2λ=x ⇒ λ=x/2

X=2y

x-2y = 0 → (4)

Solve 3 and 4

X=100

Y=50

Consider n-1= 2-1 =1 Principal mirror

Here f= xy

H= x+2y-200

So, $\Delta 3 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} = 0$

Constrained external problem with more than one equality constraint:

The general form of the non linear programming problem having n variables and m constraints (n>m) can be taken as

Optimize Z =f(x) X= (x1,x2,x3….xn)

Subject to

h(x)= 0,i=1,2,3…m

x≥0

The Lagrangean function can be taken as

L(x,λ)=f(x)-$\sum_{i=1}^{m} h^i \lambda\, h^i(x)$

Where λ_i, i=1,2,3…….. n are lagrangean multiplier

Assuming that the functions L(x, λ), f(x) and h'(x) are partially differentiable with respect to x and λ. The necessary conditions for optimum solution are

$\frac{\partial L}{\partial x_j}$=0, j=1,2…n

$\frac{\partial L}{\partial \lambda_j}$=0, i=1,2,3….m

The sufficient conditions for the stationary point to be a maximum or minimum are obtained by evaluating the principal minors of the **"Bordered Hessian matrix"**

$H^B = \begin{bmatrix} O & P \\ P^T & Q \end{bmatrix}_{(m+n)X(m+n)}$

Where O is an maximum null matrix and Q is

$$Q=\begin{bmatrix} \frac{\partial^2 z}{\partial x_1^2} & \frac{\partial^2 z}{\partial x1x2} & \frac{\partial^2 z}{\partial x1xn} \\ \frac{\partial^2 z}{\partial x2x1} & \frac{\partial^2 z}{\partial x_2^2} & \frac{\partial^2 z}{\partial x2xn} \\ \frac{\partial^2 z}{\partial x2xn} & \frac{\partial^2 z}{\partial xnx2} & \frac{\partial^2 z}{\partial x_n^2} \end{bmatrix}$$

$$P=\begin{pmatrix} h_1^1(x) & h_2^1(x) & h_1^n(x) \\ h_2^2(x) & h_2^2(x) & h_1^n(x) \\ h_1^m(x) & h_2^m(x) & h_n^m(x) \end{pmatrix}$$

Let (X^*, λ^*) be the stationary point for the function of L(x,λ) and $H^{B}*$ be the corresponding bordered hessian matrix. The sufficient but not necessary condition for the maxima or minima is determined by the signs of the last (n-m) principal minor of $H^{B}*$ starting with principal minor of order of 2m+1.

Now

X* maximizes L if the last (n-m) principal minor from an alternate sign pattern with $(-1)^{m+n}$ and

X* minimizes L if the last (n-m) principal minor from an alternate sign pattern with $(-1)^{m}$

Example problem : Solve the non-linear programming problem by Lagrangean multiplier method

Minimize z= $x_1^2+x_2^2+x_3^2$

Subject to constraints

$x_1+x_2+3x_3=2$

$5x_1+2x_2+x_3=5$

$x_1,x_2,x_3 \geq 0$

Solution:

Let f(X) = $x_1^2+x_2^2+x_3^2$, X= (x_1,x_2,x_3)

h′(X)= $x_1+x_2+3x_3-2$

h′′(X)= $5x_1+2x_2+x_3-5$

$x_1,x_2,x_3 \geq 0$

The Lagrangean function

L(X,λ)= f(X)- λ1 h′(X)- λ2h′′(X)

L= $x_1^2+x_2^2+x_3^2$- λ $(x_1+x_2+3x_3 -2)$ – λ2$(5x_1+2x_2+x_3-5)$

The stationary point (X*, λ*) is given by the following necessary conditions:

∂L/∂x1= $2x_1$- λ1-5 λ2=0 -----------(1)

∂L/∂x2= $2x_2$- λ1-2 λ2=0 ------------(2)

∂L/∂x3= $2x_2$-3λ1- λ2=0 ------------(3)

∂L/∂λ1= $-(x_1+x_2+3x_3-2)$-0 ------------(4)

∂L/∂λ2= $-(5x_1+2x_2+x_3-5)$=0 ------------(5)

from (1)
x_1= (λ1+5 λ2)/2 ------------(6)
from (2)
x_2= (λ1+2 λ2)/2 ------------(7)
from (3)
x_3= (3λ1+ λ2)/2 ------------(8)
Using (7),(8)
11λ+10 λ2=4 ------------(9)
11 λ1+33 λ2=11 -------------(10)
λ2=7/23
λ1=2/23
Using bordered hessian matrix n=3, m=2gives n-m=1; For minimization the sign $(-1)^m=(-1)^2$ is + so the solution is
Min = Z= 0.857, x_1= 37/46; x_2=8/23; x_3=13/46.

KUHN – TUCKER CONDITIONS – SIMPLE PROBLEM

KUHN TUCKER CONDITIONS – KKT
Constrained external problem with inequality constraints[KUHN TUCKER CONDITIONS – KKT]
MAXIMIZATION PROBLEM
Maximize z= f(x)
Subject to
$g(x) \leq b$ ………….(1)
$x \geq 0$, $x=(x_1,x_2,x_3\ldots\ldots x_n)$
Let h(x)= g(x)-b then $h(x) \leq 0$ from (1)
First the inequality constraint is changed to equality constraint type by introducing a slack variable S in the form of S^2 to ensure the non – negativity.

Thus the constraint can be expressed as $h(x)+S^2=0$ and the NLPP can be expressed in the form
Max Z=f(x)
Subject to
$h(x)+S^2=0$
$x \geq 0$

Construct the lagrangean function
$L(x,s,\lambda)=f(x)-\frac{\partial h}{\partial xj}=0$ ------(1) where j=1,2,3….n
$\frac{\partial l}{\partial \lambda}$ = -[h(x)+s²] = 0 --------(2)
$\frac{\partial l}{\partial s}$ = $-2s\lambda=0$ --------(3)

From (3) we have either s=0 or λ=0
If s=0 then from (2) we have, h(x) =0
Either λ=0 or h(x)=0
Ie, λ h(x) =0 ----------(4)
From (2) again we have,
h(x)= $-s^2$ =-ve
h(x)≤0 ----------(5)
Thus the necessary condition is summarized as
$\frac{\partial f}{\partial xj}$ -λ$\frac{\partial h}{\partial xj}$ =0, j=1,2…n ------------(I)
λ h(x)=0 -----(II)
h(x))≤0 ------(III)
λ≥0
These necessary conditions are called Kuhn-Tucker or Krush Kuhn Tucker conditions.

MINIMISATION PROBLEM:
Minimize
Z=f(x) , x= (x_1,x_2,x_3….x_n)
Subject to g(x) ≥b
x≥0
This is rewritten as
Minimize
Z=f(x)
Subject to
h(x)= g(x)-b≥0
x≥0
Introducing slack variablein the form s^2 we have the problem as
Minimize Z=f(x)
Subject to
$h(x)-s^2=0$
Following the analysis similar to the one use the maximization problem Kuhn tucker conditions become
$\frac{\partial f}{\partial xj}$ -λ$\frac{\partial h}{\partial xj}$ =0, j=1,2…n
λ h(x)=0
h(x))≤0
λ≥0
For a single constraint NLPP the Kuhn tucker conditions are also sufficient conditions if
(i) f(x) is concave and h(x) is concave in the maximization problem and
(ii) both f(x) and h(x) are concave in the minimization problem.

Example:
Maximize $z= 8x_1+10x_2-x1^2-x2^2$
Subject to
$3x_1+2x_2 \leq 6$
$x_1,x_2 \geq 0$

Solution:

Let $f(x) =8x_1+10x_2-x1^2-x2^2$

$h(x)= 3x_1+2x2-6$

Kuhn Tucker conditions for the maximization problem becomes

$f(x)-\lambda\ h(x)=0$

$\lambda\ h(x))\leq 0$

$\lambda \geq 0$

that is $8-2x_1-3\ \lambda=0$ -------(1)

$10-2x_2-2\ \lambda=0$ -----------(2)

$\lambda(3x_1+2x_2-6)=0$ ------(3)

$3x_1+2x_2-6\leq 0;\ \lambda\geq 0$ -----(4)

Case(i):

$\lambda=0$

The above equations becomes,

$8-2x_1=0$ from (1)

$10-2x_2 =0$ from (2)

Hence, $x_1 =4;\ x_2= 5$

This solution is not feasible since, $3x_1+2x_2 \neq 6$ when $x_1 =4;\ x_2= 5$

Case(ii):

$\lambda\neq 0$

Then the above equations becomes

$8-2x_1-3\lambda =0$

$10-2x_2-2\lambda=0$

$3x_1-2x_2-6=0$

From the above equations

$x_1= (8-3\lambda)/2$; $x_2=5- \lambda$

Using that Max $z= 21.3$; $x_1=4/13$; $x_2=33/13$

Example:

Minimize z= $0.3x_1^2-2x_1+0.4x_2^2-2.4x_2+0.6x_1x_2+100$

Subject to

$2x_1+x_2\geq 4$, $x_1,x_2\geq 0$

Solution:

Let $f(x)= 0.3x_1^2-2x_1+0.4x_2^2-2.4x_2+0.6x_1x_2+100$ and $h(x)= 2x_1+x_2-4$

Kuhn Tucker conditions for the minimization problem becomes

$f_j(x)-\lambda h_j(x)=0$

$\lambda\ h(x))\leq 0$

$\lambda\geq 0$

Thus we have

$0.6\ x_1-2+0.6x_2-2\lambda=0$ ------(1)

$0.8\ x_1-2.4+0.6x_1-\lambda=0$ ------(2)

$\lambda(2x_1+x_2-4)=0$ ----(3)

and $2x_1+x_2-4\geq 0$ ------(4)

$x_1,x_2\geq 0$, $\lambda\geq 0$

Case(i):

$\lambda= 0$

The above equations 1 and 2 becomes

$0.6x_1+0.2x_2=2$

$0.6x_2+0.6x_1=2.4$

(ie) $3x_1+3x_2=10$

$3x1+4x_2=12$

Hence, $x_2= 2$ and $x_1=4/3$

But $2x_1+x_2 =(8/3)+2; 14/3 \geq 4$

So the solution is not feasible.

Case(ii):

$\lambda\neq 0$

The Kuhn tucker conditions becomes

$0.6x_1+0.6x_2=2+2\lambda$

$0.6x_1+0.8x_2=\lambda+2.4$

$2x_2=0.4- \lambda$

$\lambda=0.4-0.2x_2$

from (3) $2x_1+x_2=4$

$x_1= -6/7<0$; $x_2=40/7$

This solution is not feasible since, x_1**<**0; The optimal solution is given by case(ii) $x_1=4/3$; $x_2=2$

So, Z_{min} is= 0.3X(16/9)-(8/3)+04X4-2.4X2+0.6X(4/3)X2+100

$Z_{min}= 292/3$

$Z_{min}=97.33$

So the solution is **$x_1=4/3$; $x_2=2$; $Z_{min}= 97.33$**

Example:

Solve the non-linear programming problem by Kuhn-Tucker conditions.

Minimize f(x) = x12 + x22 + x32

Subject to

g1(X) = 2x1 + x2 – 5 ≤ 0

g2(X) = x1 + x2 – 2 ≤ 0

g3(X) = 1 - x1 ≤ 0

g4(X) = 2 - x2 – 5 ≤ 0

g5(X) = – x3 ≤ 0

Solution:

The Kuhn-Tucker conditions are :

$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) \leq 0$

$$(2x_1, 2x_2, 2x_3) - (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = 0$$

$\lambda_i g_i = 0$ for i = 1 to 5

$g(X) \leq 0$

Also, g(X) ≤ 0

Thus, we now have

$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \leq 0$

From the first condition.

From the second condition, we have

$2x_1 - 2\lambda_1 - \lambda_2 + \lambda_3 = 0$ → (i)

$2x_2 - \lambda_1 + \lambda_4 = 0$ → (ii)

$2x_3 - \lambda_2 + \lambda_5 = 0$ → (iii

From the third condition, we have

$\lambda_1(2x_1 + x_2 - 5) = 0$ → (iv)

$\lambda_2(x_1 + x_3 - 2) = 0$ → (v)

$\lambda_3(1 - x_1) = 0$ → (vi)

JACOBIAN METHODS

If U and V are functions of two independent variables x and y then the following determinant becomes $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called Jacobian of u and v with respect to x and y, its denoted by $\frac{\partial(u,v)}{\partial(x,y)}$ or $J\frac{u,v}{x,y}$

Now we are going to see how Jacobian method can be determined.

Example:

If x= rcosθ, y=rsinθ find $\frac{\partial(x,y)}{\partial(r,\theta)}$

Solution:

Given:

x=rcosθ y=rsinθ

$\frac{\partial x}{\partial r}$= cosθ $\frac{\partial y}{\partial r}$=sinθ

$\frac{\partial x}{\partial \theta}$= -sinθ $\frac{\partial y}{\partial \theta}$ =rcosθ

We know that $\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$$=\begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix}$$

$=r(\cos^2\theta+r\sin^2\theta)$ $[\cos^2\theta+r\sin^2\theta=1]$

=r

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$

Example:

If x=acoshϕcosθ, y=asinhϕsinθ; show that $\frac{\partial(x,y)}{\partial(\phi,\theta)}$= $\frac{a^2}{2}$[cosh2ϕ-cos2θ]

Solution:

x=asinhϕcosϕ, $\frac{\partial y}{\partial \theta}$= acoshϕsinθ

$\frac{\partial x}{\partial \theta}$=-acoshϕsinθ, $\frac{\partial y}{\partial \theta}$= asinhϕcosθ

We know that $\frac{\partial(x,y)}{\partial(\phi,\theta)}$= = $\begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$$= \begin{vmatrix} asinh\phi cos\theta & -acosh\phi sin\theta \\ acosh\phi sin\theta & asinh\phi cos\theta \end{vmatrix}$$

$$=a^2 \sinh^2\phi \sin\theta \cosh^2\theta + a^2 \cosh^2\phi \sin^2\theta$$

$$=a^2[\sinh^2\phi(1-\sin^2\theta)+(1+\sinh^2\phi)\sin^2\theta]$$

$$=a^2[\sinh^2\phi- \sinh^2\phi\sin^2\theta+\sin^2\theta+ \sinh^2\phi\sin^2\theta]$$

$$=a^2[\sinh^2\phi+ \sin^2\theta]$$

$$=a^2[\frac{cosh2\phi-1}{2} +\frac{1-cos2\theta}{2}]$$

$$=\frac{a^2}{2}[\cosh2\phi-\cos2\theta]$$

$$\frac{\partial(x,y)}{\partial(\phi,\theta)} = \frac{a^2}{2}[\cosh2\phi-\cos2\theta]$$

Example:

Find the value of jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ where u=x^2-y^2, v=2xy and x=rcosθ, y=rsinθ

Solution:

Given

u=x^2-y^2

v=2xy

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{vmatrix}$$

$$= r\cos^2\theta+r\sin^2\theta$$

$$=r(\cos^2\theta+\sin^2\theta)$$

$$=r$$