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DEPARTMENT OF MECHANICAL ENGINEERING

ME3491 THEORY OF MACHINES

COURSE MATERIAL

UNIT – 4 - FORCE ANALYSIS

1.1 INTRODUCTION

The subject Dynamics of Machines may be defined as that branch of Engineering-science, which deals with the study of relative motion between the various parts of a machine, and forces which act on them. The knowledge of this subject is very essential for an engineer in designing the various parts of a machine.

A machine is a device which receives energy in some available form and utilises it to do some particular type of work.

If the acceleration of moving links in a mechanism is running with considerable amount of linear and/or angular accelerations, inertia forces are generated and these inertia forces also must be overcome by the driving motor as an addition to the forces exerted by the external load or work the mechanism does.

1.2 NEWTON' S LAW :

First Law

Everybody will persist in its state of rest or of uniform motion (constant velocity) in a straight line unless it is compelled to change that state by forces impressed on it. This means that in the absence of a non-zero net force, the center of mass of a body either is at rest or moves at a constant velocity.

Second Law

A body of mass *m* subject to a force **F** undergoes an acceleration **a** that has the same direction as the force and a magnitude that is directly proportional to the force and inversely proportional to the mass, i.e., **F** = *m***a**. Alternatively, the total force applied on a body is equal to the time derivative of linear momentum of the body.

Third Law

The mutual forces of action and reaction between two bodies are equal, opposite and collinear. This means that whenever a first body exerts a force **F** on a second body, the second body exerts a force −**F** on the first body. **F** and −**F** are equal in magnitude and opposite in direction. This law is sometimes referred to as the *action-reaction law*, with **F** called the "action" and −**F** the "reaction"

1.3 TYPES OF FORCE ANALYSIS:

- Equilibrium of members with two forces
- Equilibrium of members with three forces
- Equilibrium of members with two forces and torque Equilibrium of members with two couples.
  Equilibrium of members with four forces.

1.3.1 Principle of Super Position:

Sometimes the number of external forces and inertial forces acting on a mechanism are too much for graphical solution. In this case we apply the method of superposition. Using superposition the entire system is broken up into (n) problems, where n is the number of forces, by considering the external and inertial forces of each link individually. Response of a linear system to several forces acting simultaneously is equal to the sum of responses of the system to the forces individually. This approach is useful because it can be performed by graphically.

1.3.2 Free Body Diagram:

A free body diagram is a pictorial representation often used by physicists and engineers to analyze the forces acting on a body of interest. A free body diagram shows all forces of all types acting on this body. Drawing such a diagram can aid in solving for the unknown forces or the equations of motion of the body. Creating a free body diagram can make it easier to understand the forces, and torques or moments, in relation to one another and suggest the proper concepts to apply in order to find the solution to a problem. The diagrams are also used as a conceptual device to help identify the internal forces—for example, shear forces and bending moments in beams—which are developed within structures.

1.4 DYNAMIC ANALYSIS OF FOUR BAR MECHANISM:

A **four-bar linkage** or simply a **4-bar** or **four-bar** is the simplest movable linkage. It consists of four rigid bodies (called bars or links), each attached to two others by single joints or pivots to form closed loop. Fourbars are simple mechanisms common in mechanical engineering machine design and fall under the study of kinematics.

- Dynamic Analysis of Reciprocating engines.
- Inertia force and torque analysis by neglecting weight of connecting rod.
- Velocity and acceleration of piston.
- Angular velocity and Angular acceleration of connecting rod.
- Force and Torque Analysis in reciprocating engine neglecting the weight of connecting rod.
- Equivalent Dynamical System
- Determination of two masses of equivalent dynamical system

The inertia force is an imaginary force, which when acts upon a rigid body, brings it in an equilibrium position. It is numerically equal to the accelerating force in magnitude, but ***opposite*** in direction. Mathematically,

Inertia force = – Accelerating force = $-m.a$

where m = Mass of the body, and

a = Linear acceleration of the centre of gravity of the body.

Similarly, the inertia torque is an imaginary torque, which when applied upon the rigid body, brings it in equilib-rium position. It is equal to the accelerating couple in magni-tude but ***opposite*** in direction.

1.4.1 D-Alembert's Principle

Consider a rigid body acted upon by a system of forces. The system may be reduced to a single resultant force acting on the body whose magnitude is given by the product of the mass of the body and the linear acceleration of the centre of mass of the body. According to Newton's second law of motion,

$F = m.a$

F = Resultant force acting on the body, m

Mass of the body, and

Linear acceleration of the centre of mass of the

a body.

The equation ***(i)*** may also be written as:

$F - m.a = 0$

A little consideration will show, that if the quantity $- m.a$ be treated as a force, equal, opposite and with the same line of action as the resultant force F, and include this force with the system of forces of which F is the resultant, then the complete system of forces will be in equilibrium. This principle is known as ***DAlembert's principle.*** The equal and opposite force $- m.a$ is known as ***reversed effective force*** or the ***inertia force*** (briefly written as F_I). The equation ***(ii)*** may be written as

$+ F_I = 0$...***(iii)***

Thus, D-Alembert's principle states that ***the resultant force acting on a body together with the reversed effective force (or inertia force), are in equilibrium.***

This principle is used to reduce a dynamic problem into an equivalent static problem.

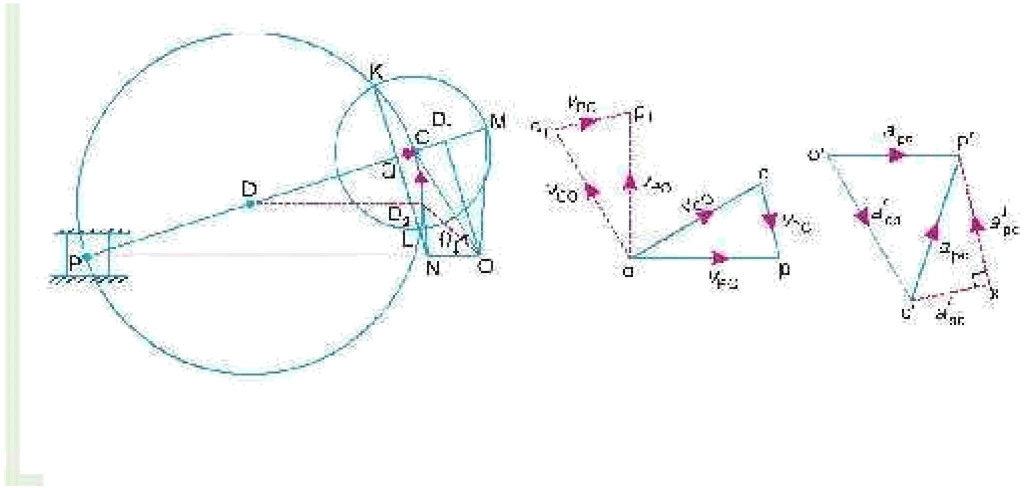
1.4.2 Velocity and Acceleration of the Reciprocating Parts in Engines

The velocity and acceleration of the reciprocating parts of the steam engine or internal combustion engine (briefly called as I.C. engine) may be determined by graphical method or analytical method. The velocity and acceleration, by graphical method, may be determined by one of the following constructions: **1.** Klien's construction, **2.** Ritterhaus's construction, and **3.** Bennett's construction.

We shall now discuss these constructions, in detail, in the following pages.

1.5 KLIEN'S CONSTRUCTION

Let OC be the crank and PC the connecting rod of a reciprocating steam engine, as shown in Fig. 15.2 (*a*). Let the crank makes an angle θ with the line of stroke PO and rotates with uniform angular velocity ω rad/s in a clockwise direction. The Klien's velocity and acceleration diagrams are drawn as discussed below:



(*a*)

Klien's acceleration diagram. (*b*) Velocity diagram. (*c*) Acceleration diagram.

Fig. 15.2. Klien's construction.

1.5.1 Klien's velocity diagram

First of all, draw *OM* perpendicular to *OP*; such that it intersects the line *PC* produced at *M*. The triangle *OCM* is known as ***Klien's velocity diagram***. In this triangle *OCM*,

OM may be regarded as a line perpendicular to *PO*,

CM may be regarded as a line parallel to *PC*, and ...(Q It is the same line.)

CO may be regarded as a line parallel to *CO*.

We have already discussed that the velocity diagram for given configuration is a triangle *ocp* as shown in Fig. 15.2 (*b*). If this triangle is revolved through 90°, it will be a triangle $oc_1\ p_1$, in which oc_1 represents v_{CO} (*i.e.* velocity of *C* with respect to *O* or velocity of crank pin *C*) and is paralel to *OC*, op_1 represents v_{PO} (*i.e.* velocity of *P* with respect to *O* or velocity of cross-head or piston *P*) and is perpendicular to *OP*, and c_1p_1 represents v_{PC} (*i.e.* velocity of *P* with respect to *C*) and is parallel to *CP*.

Thus, we see that by drawing the Klien's velocity diagram, the velocities of various points may be obtained without drawing a separate velocity diagram.

1.5.2 Klien's acceleration diagram

The Klien's acceleration dia-gram is drawn as discussed below:

First of all, draw a circle with *C* as centre and *CM* as radius.

Draw another circle with *PC* as diameter. Let this circle intersect the previous circle at *K* and *L*.

Join *KL* and produce it to intersect *PO* at *N*. Let *KL* intersect *PC* at *Q*.

This forms the quadrilateral *CQNO*, which is known as ***Klien's acceleration diagram*.**

We have already discussed that the acceleration diagram for the given configuration is as shown in Fig. 15. 2 (*c*). We know that (*i*) $o'c'$ represents a^r_{CO} (*i.e.* radial component of the acceleration of crank pin *C* with respect to *O*) and is parallel to *CO*; (*ii*) $c'x$ represents a^r_{PC} (*i.e.* radial component of the acceleration of crosshead or piston *P* with respect to crank pin *C*) and is parallel to *CP* or *CQ*;

xp' represents a^t_{PC} (*i.e.* tangential component of the acceleration of *P* with respect to *C*) and is parallel to *QN* (because *QN* is perpendicular to *CQ*); and

$o'p'$ represents a_{PO} (*i.e.* acceleration of *P* with respect to *O* or the acceleration of piston *P*) and is parallel to *PO* or *NO*.

A little consideration will show that the quadrilateral $o'c'x\,p'$ [Fig. 15.2 (*c*)] is similar to quadrilateral *CQNO* [Fig. 15.2 (*a*)]. Therefore,

$$\frac{o'c'}{OC} = \frac{c'x}{CQ} = \frac{xp'}{QN} = \frac{o'p'}{NO} = \omega^2 \text{ (a constant)}$$

or

$$\frac{a^r_{CO}}{OC} = \frac{a^r_{PC}}{CQ} = \frac{a^t_{PC}}{QN} = \frac{a_{PO}}{NO} = \omega^2$$

$\therefore$

$$a^r_{CO} = \omega^2 \times OC;\ a^r_{PC} = \omega^2 \times CQ$$

$$a^t_{PC} = \omega^2 \times QN;\ \text{and } a_{PO} = \omega^2 \times NO$$

Thus we see that by drawing the Klien's acceleration diagram, the acceleration of various points may be obtained without drawing the separate acceleration diagram.

1.6 SOLVED PROBLEMS

1. *The crank and connecting rod of a reciprocating engine are 200 mm and 700 mm respectively. The crank is rotating in clockwise direction at 120 rad/s. Find with the help of Klein's construction:* **1.** *Velocity and acceleration of the piston,* **2.** *Velocity and acceleration of the mid point of the connecting rod, and* **3.** *Angular velocity and angular acceleration of the connecting rod, at the instant when the crank is at 30° to I.D.C.* (*inner dead centre*).

Solution. Given: $OC = 200$ mm = 0.2 m ; $PC = 700$ mm = 0.7 m ; $\omega = 120$ rad/s

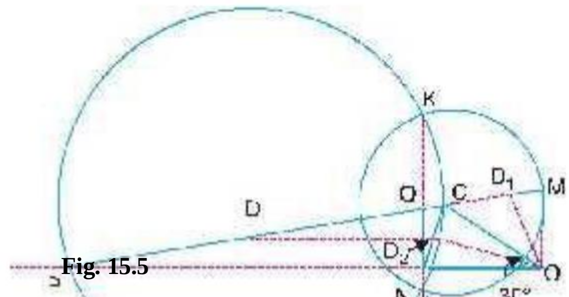


Fig. 15.5

The Klein's velocity diagram *OCM* and Klein's acceleration diagram *CQNO* as shown in Fig. 15.5 is drawn to some suitable scale, in the similar way as discussed in Art. 15.5. By measurement, we find that

$OM = 127$ mm = 0.127 m ; $CM = 173$ mm = 0.173 m ; $QN = 93$ mm = 0.093 m ; $NO = 200$ mm 0.2 m

Velocity and acceleration of the piston

We know that the velocity of the piston *P*,

$$v_P = \omega \times OM = 120 \times 0.127 = 15.24 \text{ m/s}$$ **Ans.** and

acceleration of the piston *P*,

$$a_P = \omega^2 \times NO = (120)^2 \times 0.2 = 2880 \text{ m/s}^2$$ **Ans.**

Velocity and acceleration of the mid-point of the connecting rod

In order to find the velocity of the mid-point *D* of the connecting rod, divide *CM* at D_1 in the same ratio as *D* divides *CP*. Since *D* is the mid-point of *CP*, therefore D_1 is the mid-point of *CM*, *i.e.* $CD_1 = D_1M$. Join OD_1. By measurement,

$$OD_1 = 140 \text{ mm} = 0.14 \text{ m}$$

$\therefore$ Velocity of *D*, $v_D = \omega \times OD_1 = 120 \times 0.14 = 16.8$ m/s **Ans.**

In order to find the acceleration of the mid-point of the connecting rod, draw a line DD_2 parallel to the line of stroke *PO* which intersects *CN* at D_2. By measurement,

$$OD_2 = 193 \text{ mm} = 0.193 \text{ m}$$

Acceleration of *D*,

$$a_D = \omega^2 \times OD_2 = (120)^2 \times 0.193 = 2779.2 \text{ m/s}^2$$ **Ans.**

Angular velocity and angular acceleration of the connecting rod

We know that the velocity of the connecting rod *PC* (*i.e.* velocity of *P* with respect to *C*), $v_{PC} = \omega \times CM = 120 \times 0.173 = 20.76$ m/s

∴ Angular acceleration of the connecting rod PC,

$$\omega_{PC} = \frac{v_{PC}}{PC} = \frac{20.76}{0.7} = 29.66 \text{ rad/s} \quad \textbf{Ans.}$$

We know that the tangential component of the acceleration of P with respect to C,

$$a^t_{PC} = \omega^2 \times QN = (120)^2 \times 0.093 = 1339.2 \text{ m/s}^2$$

∴ Angular acceleration of the connecting rod PC,

$$\alpha_{PC} = \frac{a^t_{PC}}{PC} = \frac{1339.2}{0.7} = 1913.14 \text{ rad/s}^2 \quad \textbf{Ans.}$$

1.7 APPROXIMATE ANALYTICAL METHOD FOR VELOCITY AND ACCELERATION OF THE PISTON

Consider the motion of a crank and connecting rod of a reciprocating steam engine as shown in Fig. 15.7. Let *OC* be the crank and *PC* the connecting rod. Let the crank rotates with angular velocity of ω rad/s and the crank turns through an angle θ from the inner dead centre (briefly written as I.D.C). Let *x* be the displacement of a reciprocating body *P* from I.D.C. after time *t* seconds, during which the crank has turned through an angle θ .

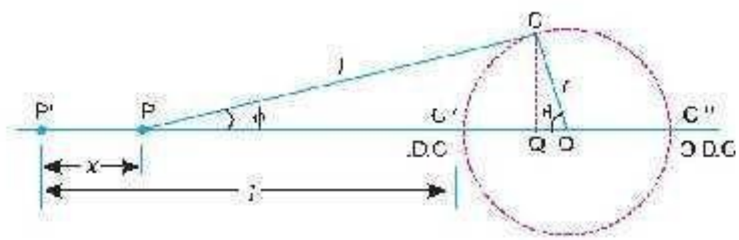


Fig. 15.7. Motion of a crank and connecting rod of a reciprocating steam engine.

Let l = Length of connecting rod between the centres,

r = Radius of crank or crank pin circle,

= Inclination of connecting rod to the line of stroke *PO*, and n = Ratio of length of connecting rod to the radius of crank = l/r.

Velocity of the piston:

From the geometry of Fig. 15.7,

$$x = P'P = OP' - OP = (P'C' + C'O) - (PQ + QO)$$

$$= (l + r) - (l \cos\phi + r \cos\theta) \qquad \left(\begin{matrix} \because \; PQ = l\cos\phi, \\ \text{and } QO = r\cos\theta \end{matrix}\right)$$

$$= r(1 - \cos\theta) + l(1 - \cos\phi) = r\left[(1 - \cos\theta) + \frac{l}{r}(1 - \cos\phi)\right]$$

$$= r[(1 - \cos\theta) + n(1 - \cos\phi)] \qquad ...(i)$$

From triangles CPQ and CQO,

$$CQ = l\sin\phi = r\sin\theta \text{ or } l/r = \sin\theta/\sin\phi$$

$$\therefore \qquad n = \sin\theta/\sin\phi \text{ or } \sin\phi = \sin\theta/n \qquad ...(ii)$$

We know that, $\cos\phi = \left(1 - \sin^2\phi\right)^{\frac{1}{2}} = \left(1 - \frac{\sin^2\theta}{n^2}\right)^{\frac{1}{2}}$

Expanding the above expression by binomial theorem, we get

$$\cos\phi = 1 - \frac{1}{2} \times \frac{\sin^2\theta}{n^2} + ... \qquad \text{...(Neglecting higher terms)}$$

or

$$1 - \cos\phi = \frac{\sin^2\theta}{2n^2} \qquad ...(iii)$$

Substituting the value of $(1 - \cos\phi)$ in equation (*i*), we have

$$x = r\left[(1 - \cos\theta) + n \times \frac{\sin^2\theta}{2n^2}\right] = r\left[(1 - \cos\theta) + \frac{\sin^2\theta}{2n}\right] \qquad ..(iv)$$

Differentiating equation (*iv*) with respect to θ,

$$\frac{dx}{d\theta} = r\left[\sin\theta + \frac{1}{2n} \times 2\sin\theta.\cos\theta\right] = r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right) \qquad ...(v)$$

$$(\because 2\sin\theta.\cos\theta = \sin 2\theta)$$

∴ Velocity of P with respect to O or velocity of the piston P,

$$v_{PO} = v_P = \frac{dx}{dt} = \frac{dx}{d\theta} \times \frac{d\theta}{dt} = \frac{dx}{d\theta} \times \omega$$

...(∵ Ratio of change of angular velocity = $d\theta/dt = \omega$)

Substituting the value of $dx/d\theta$ from equation (*v*), we have

$$v_{PO} = v_P = \omega r\left(\sin\theta + \frac{\sin 2\theta}{2n}\right) \qquad ...(vi)$$

Acceleration of the piston:

Since the acceleration is the rate of change of velocity, therefore acceleration of the piston *P*,

$$a_P = \frac{dv_P}{dt} = \frac{dv_P}{d\theta} \times \frac{d\theta}{dt} = \frac{dv_P}{d\theta} \times \omega$$

Differentiating equation (*vi*) with respect to θ,

$$\frac{dv_P}{d\theta} = \omega . r \left[\cos\theta + \frac{\cos 2\theta \times 2}{2n} \right] = \omega . r \left[\cos\theta + \frac{\cos 2\theta}{n} \right]$$

Substituting the value of $\frac{dv_P}{d\theta}$ in the above equation, we have

$$a_P = \omega . r \left[\cos\theta + \frac{\cos 2\theta}{n} \right] \times \omega = \omega^2 . r \left[\cos\theta + \frac{\cos 2\theta}{n} \right] \quad ...(vii)$$

1.8 ANGULAR VELOCITY AND ACCELERATION OF THE CONNECTING ROD

Consider the motion of a connecting rod and a crank as shown in Fig. 15.7.From the geometry of the figure, we find that

$$CQ = l \sin\phi = r \sin\theta$$

$$\therefore \quad \sin\phi = \frac{r}{l} \times \sin\theta = \frac{\sin\theta}{n} \quad ...\left(\because n = \frac{l}{r}\right)$$

Differentiating both sides with respect to time *t*,

$$\cos\phi \times \frac{d\phi}{dt} = \frac{\cos\theta}{n} \times \frac{d\theta}{dt} = \frac{\cos\theta}{n} \times \omega \quad \left(\because \frac{d\theta}{dt} = \omega\right)$$

Since the angular velocity of the connecting rod *PC* is same as the angular velocity of point *P* with respect to *C* and is equal to $d\phi/dt$, therefore angular velocity of the connecting rod

$$\omega_{PC} = \frac{d\phi}{dt} = \frac{\cos\theta}{n} \times \frac{\omega}{\cos\phi} = \frac{\omega}{n} \times \frac{\cos\theta}{\cos\phi}$$

We know that, $\cos\phi = (1 - \sin^2\phi)^{\frac{1}{2}} = \left(1 - \frac{\sin^2\theta}{n^2}\right)^{\frac{1}{2}} \quad ...\left(\because \sin\phi = \frac{\sin\theta}{n}\right)$

$$\therefore \quad \omega_{PC} = \frac{\omega}{n} \times \frac{\cos\theta}{\left(1 - \frac{\sin^2\theta}{n^2}\right)^{\frac{1}{2}}} = \frac{\omega}{n} \times \frac{\cos\theta}{\frac{1}{n}(n^2 - \sin^2\theta)^{1/2}}$$

$$= \frac{\omega \cos\theta}{(n^2 - \sin^2\theta)^{1/2}} \quad ...(i)$$

Angular acceleration of the connecting rod *PC*,

$$\alpha_{PC} = \text{Angular acceleration of } P \text{ with respect to } C = \frac{d(\omega_{PC})}{dt}$$

We know that

$$\frac{d(\omega_{PC})}{dt} = \frac{d(\omega_{PC})}{d\theta} \times \frac{d\theta}{dt} = \frac{d(\omega_{PC})}{d\theta} \times \omega \qquad ...(ii)$$

$$...(\because d\theta / dt = \omega)$$

Now differentiating equation (i), we get

$$\frac{d(\omega_{PC})}{d\theta} = \frac{d}{d\theta}\left[\frac{\omega \cos\theta}{(n^2 - \sin^2\theta)^{1/2}}\right]$$

$$= \omega\left[\frac{(n^2 - \sin^2\theta)^{1/2}(-\sin\theta)] - [(\cos\theta) \times \frac{1}{2}(n^2 - \sin^2\theta)^{-1/2} \times -2\sin\theta\cos\theta}{n^2 - \sin^2 0}\right]$$

$$= \omega\left[\frac{(n^2 - \sin^2\theta)^{1/2}(-\sin\theta) + (n^2 - \sin^2\theta)^{-1/2}\sin\theta\cos^2\theta}{n^2 - \sin^2\theta}\right]$$

$$= -\omega\sin\theta\left[\frac{(n^2 - \sin^2\theta)^{1/2} - (n^2 - \sin^2\theta)^{-1/2}\cos^2\theta}{n^2 - \sin^2\theta}\right]$$

$$= -\omega\sin\theta\left[\frac{(n^2 - \sin^2\theta) - \cos^2\theta}{(n^2 - \sin^2\theta)^{3/2}}\right] \qquad ...[\text{Dividing and multiplying by } (n^2 - \sin^2 0)^{1/2}]$$

$$= \frac{-\omega\sin\theta}{(n^2 - \sin^2\theta)^{3/2}}\left[n^2 - (\sin^2\theta + \cos^2\theta)\right] = \frac{-\omega\sin\theta(n^2 - 1)}{(n^2 - \sin^2\theta)^{3/2}}$$

$$...(\because \sin^2\theta + \cos^2\theta = 1)$$

$$\therefore \alpha_{PC} = \frac{d(\omega_{PC})}{d\theta} \times \omega = \frac{-\omega^2\sin\theta(n^2 - 1)}{(n^2 - \sin^2\theta)^{3/2}} \qquad ...[\text{From equation } (ii)] \qquad ...(iii)$$

The negative sign shows that the sense of the acceleration of the connecting rod is such that it tends to reduce the angle φ..

2.In a slider crank mechanism, the length of the crank and connecting rod are 150 mm and 600 mm respectively. The crank position is 60° from inner dead centre. The crank shaft speed is 450 r.p.m. (*clockwise*). *Using analytical method, determine: 1. Velocity and acceleration of the slider, and 2. Angular velocity and angular acceleration of the connecting rod.*

Solution. Given : r = 150 mm = 0.15 m ; l = 600 mm = 0.6 m ; θ = 60°; N = 400 r.p.m or = π × 450/60 = 47.13 rad/s

Velocity and acceleration of the slider

We know that ratio of the length of connecting rod and crank, n =

l / r = 0.6 / 0.15 = 4

∴ Velocity of the slider,

$$v_P = \omega r \left(\sin\theta + \frac{\sin 2\theta}{2n} \right) = 47.13 \times 0.15 \left(\sin 60^\circ + \frac{\sin 120^\circ}{2 \times 4} \right) \text{m/s}$$

$$= 6.9 \text{ m/s} \quad \textbf{Ans.}$$

and acceleration of the slider,

$$a_P = \omega^2 r \left(\cos\theta + \frac{\cos 2\theta}{n} \right) = (47.13)^2 \times 0.15 \left(\cos 60^\circ - \frac{\cos 120^\circ}{4} \right) \text{m/s}^2$$

$$= 124.94 \text{ m/s}^2 \quad \textbf{Ans.}$$

2. Angular velocity and angular acceleration of the connecting rod

We know that angular velocity of the connecting rod,

$$\omega_{PC} = \frac{\omega \cos\theta}{n} = \frac{47.13 \times \cos 60^\circ}{4} = 5.9 \text{ rad/s} \quad \textbf{Ans.}$$

and angular acceleration of the connecting rod,

$$\alpha_{PC} = \frac{\omega^2 \sin\theta}{n} = \frac{(47.13)^2 \times \sin 60^\circ}{4} = 481 \text{ rad/s}^2 \quad \textbf{Ans.}$$

1.9 FORCES ON THE RECIPROCATING PARTS OF AN ENGINE, NEGLECTING THE WEIGHT OF THE CONNECTING ROD

The various forces acting on the reciprocating parts of a horizontal engine are shown in Fig. 15.8. The expressions for these forces, neglecting the weight of the connecting rod, may be derived as discussed below :

Piston effort. It is the net force acting on the piston or crosshead pin, along the line of stroke. It is denoted by F_P in Fig. 15.8.

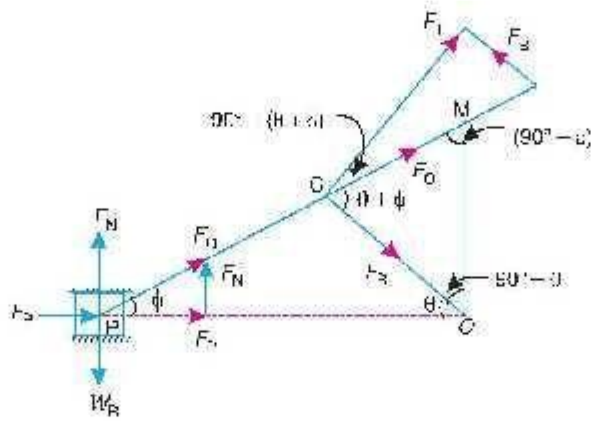


Fig. 15.8. Forces on the reciprocating parts of an engine.

Let m_R = Mass of the reciprocating parts, *e.g.* piston, crosshead pin or gudgeon pin etc., in kg, and

W_R = Weight of the reciprocating parts in newtons = $m_R.g$ We know that acceleration of the reciprocating parts,

$$a_R = a_P = \omega^2 . r \left(\cos\theta + \frac{\cos 2\theta}{n} \right)$$

Accelerating force or inertia force of the reciprocating parts,

$$F_I = m_R \cdot a_R = m_R . \omega^2 . r \left(\cos\theta + \frac{\cos 2\theta}{n} \right)$$

It may be noted that in a horizontal engine, the reciprocating parts are accelerated from rest, during the latter half of the stroke (*i.e.* when the piston moves from inner dead centre to outer dead centre). It is, then, retarded during the latter half of the stroke (*i.e.* when the piston moves from outer dead centre to inner dead centre). The inertia force due to the acceleration of the reciprocating parts, opposes the force on the piston due to the difference of pressures in the cylinder on the two sides of the piston. On the other hand, the inertia force due to retardation of the reciprocating parts, helps the force on the piston.

Therefore,

Piston effort, F_P = Net load on the piston $\mp$ Inertia force

$= F_L \mp F_I$ ...(Neglecting frictional resistance)

$= F_L \mp F_I - R_F$ ...(Considering frictional resistance)

where R_F = Frictional resistance.

The −ve sign is used when the piston is accelerated, and +ve sign is used when the piston is retarded.

In a double acting reciprocating steam engine, net load on the piston,

$$F_L = p_1 A_1 - p_2 A_2 = p_1 A_1 - p_2 (A_1 - a)$$

where p_1, A_1 = Pressure and cross-sectional area on the back end side of the piston,

p_2, A_2 = Pressure and cross-sectional area on the crank end side of the piston,

a = Cross-sectional area of the piston rod.

Force acting along the connecting rod. It is denoted by F_Q in Fig. 15.8. From the geom-etry of the figure, we find that

$$F_Q = \frac{F_P}{\cos\varphi}$$

We know that $\cos\phi = \sqrt{1 - \frac{\sin^2\theta}{n^2}}$

$$\therefore \quad F_Q = \frac{F_P}{\sqrt{1 - \frac{\sin^2\theta}{n^2}}}$$

Thrust on the sides of the cylinder walls or normal reaction on the guide bars. It is denoted by F_N in Fig. 15.8. From the figure, we find that

$$F_N = F_Q \sin\phi = \frac{F_P}{\cos\phi} \times \sin\phi = F_P \tan\phi \qquad \ldots\left[\because F_Q = \frac{F_P}{\cos\phi}\right]$$

Crank-pin effort and thrust on crank shaft bearings. The force acting on the connecting rod F_Q may be resolved into two components, one perpendicular to the crank and the other along the crank. The component of F_Q perpendicular to the crank is known as ***crank-pin effort*** and it is ***denoted by F_T*** in Fig. 15.8. The component of F_Q along the crank produces a thrust on the crank shaft bearings and it is denoted by F_B in Fig. 15.8. Resolving F_Q perpendicular to the crank,

$$F_T = F_Q \sin(\theta + \phi) = \frac{F_P}{\cos\phi} \times \sin(\theta + \phi)$$

and resolving F_Q along the crank,

$$F_B = F_Q \cos(\theta + \phi) = \frac{F_P}{\cos\phi} \times \cos(\theta + \phi)$$

Crank effort or ***turning moment or torque on the crank shaft.*** The product of the crank-pin effort (F_T) and the crank pin radius (r) is known as ***crank effort*** or ***turning moment*** or ***torque on the crank shaft***. Mathematically, Crank effort, $T = F_T \times r = \frac{F_P \sin(\theta + \phi)}{\cos \phi} \times r$

$$= \frac{F_P (\sin\theta \cos\phi + \cos\theta \sin\phi)}{\cos\phi} \times r$$

$$= F_P \left(\sin\theta + \cos\theta \times \frac{\sin\phi}{\cos\phi} \right) \times r$$

$$= F_P (\sin\theta + \cos\theta \tan\phi) \times r \quad \text{...(i)}$$

We know that $l \sin\phi = r \sin\theta$

$$\sin\phi = \frac{r}{l} \sin\theta = \frac{\sin\theta}{n} \quad \text{...}\left(\because n = \frac{l}{r}\right)$$

and $$\cos\phi = \sqrt{1 - \sin^2\phi} = \sqrt{1 - \frac{\sin^2\theta}{n^2}} = \frac{1}{n}\sqrt{n^2 - \sin^2\theta}$$

$\therefore$ $$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{\sin\theta}{n} \times \frac{n}{\sqrt{n^2 - \sin^2\theta}} = \frac{\sin\theta}{\sqrt{n^2 - \sin^2\theta}}$$

Substituting the value of $\tan\phi$ in equation *(i)*, we have crank effort,

$$T = F_P \left(\sin\theta + \frac{\cos\theta \sin\theta}{\sqrt{n^2 - \sin^2\theta}} \right) \times r$$

$$= F_P \times r \left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}} \right) \quad \text{...(ii)}$$

...($\because 2\cos\theta \sin\theta = \sin 2\theta$)

The crank-pin circle radius of a horizontal engine is 300 mm. The mass of the reciprocating parts is 250 kg. When the crank has travelled 60° from I.D.C., the difference between the driving and the back pressures is 0.35 N/mm² length between centre the cylinder bore is 0.5 m. If the engine runs at 250 r.p.m. and if the effect of piston ro neglected, calculate : ***1.*** *bars,* ***2.*** *thrust in the connecting rod,* ***3.*** *tangential force on the cr the crank shaft.*

The connecting rod pressure on slide pin, and ***4.*** *Turning moment of the shaft*

Solution. Given: r = 300 mm = 0.3 m ; m_R = 250 kg; θ = 60°; $p_1 - p_2$ = 0.35 N/mm²; l = 1.2 m ; D = 0.5 m = 500 mm ; N = 250 r.p.m. or ω = 2 π × 250/60 = 26.2 rad/s

First of all, let us find out the piston effort (F_P).

We know that net load on the piston,

$$F_L = (p_1 - p_2)\frac{\pi}{4} \times D^2 = 0.35 \times \frac{\pi}{4}(500)^2 = 68\,730 \text{ N}$$

...($\because$ Force = Pressure × Area)

Ratio of length of connecting rod and crank,

$$n = l / r = 1.2 / 0.3 = 4$$

and accelerating or inertia force on reciprocating parts,

$$F_I = m_R \cdot \omega^2 r \left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$

$$= 250\,(26.2)^2\; 0.3 \left(\cos 60° + \frac{\cos 120°}{4}\right) = 19\,306 \text{ N}$$

∴ Piston effort, $F_P = F_L - F_I = 68\,730 - 19\,306 = 49\,424 \text{ N} = 49.424 \text{ kN}$

Pressure on slide bars

Let φ = Angle of inclination of the connecting rod to the line of stroke.

We know that,

$$\sin\varphi = \frac{\sin\theta}{4} = \frac{\sin 60°}{4} = \frac{0.866}{4} = 0.2165\, n$$

$$\varphi = 12.5°$$

We know that pressure on the slide bars,

$$F_N = F_P \tan\varphi = 49.424 \times \tan 12.5° = 10.96 \text{ kN} \quad \textbf{Ans.}$$

Thrust in the connecting rod

We know that thrust in the connecting rod,

$$F_Q = \frac{F_P}{\cos\varphi} = \frac{49.424}{\cos 12.5°} = 50.62 \text{ kN} \quad \textbf{Ans.}$$

Tangential force on the crank-pin

We know that tangential force on the crank pin,

$$F_T = F_Q \sin(\theta + \varphi) = 50.62 \sin(60° + 12.5°) = 48.28 \text{ kN} \quad \textbf{Ans.}$$

Turning moment on the crank shaft

We know that turning moment on the crank shaft,

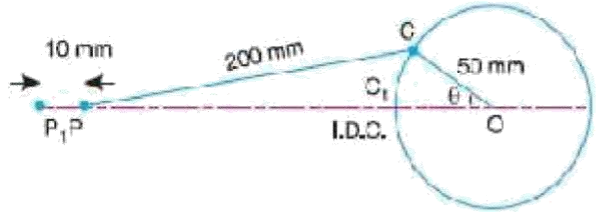
$$T = F_T \times r = 48.28 \times 0.3 = 14.484 \text{ kN-m} \quad \textbf{Ans.}$$

4. *The crank and connecting rod of a petrol engine, running at 1800 r.p.m.are 50 mm and 200 mm respectively. The diameter of the piston is 80 mm and the mass of the reciprocating parts is 1 kg. At a point during the power stroke, the pressure on the piston is 0.7 N/mm², when it has moved 10 mm from the inner dead centre. Determine :* **1.** *Net load on the gudgeon pin,* **2.** *Thrust in the connecting rod,* **3.** *Reaction between the piston and cylinder, and* **4.** *The engine speed at which the above values become zero.*

Solution. Given : $N = 1800$ r.p.m. or $\omega = 2\pi \times 1800/60 = 188.52$ rad/s ; $r = 50$ mm = 0.05 m; $l = 200$ mm ; $D = 80$ mm ; $m_R = 1$ kg ; $p = 0.7$ N/mm^2 ; $x = 10$ mm

Net load on the gudgeon pin We know that load on the piston,

$$F_L = \frac{\pi}{4} D^2 \times p = \frac{\pi}{4} \times (80)^2 \times 0.7 = 3520 \text{ N}$$



When the piston has moved 10 mm from the inner dead centre, *i.e.* when P_1P = 10 mm, the crank rotates from OC_1 to OC through an angle θ as shown in Fig. 15.10. By measurement, we find that *θ = 33°.

We know that ratio of lengths of connecting rod and crank,

$$= l/r = 200/50 = 4$$

and inertia force on the reciprocating parts,

$$F_I = m_R . a_R = m_R . \omega^2 . r \left(\cos\theta + \frac{\cos 2\theta}{n} \right)$$

$$= 1 \times (188.52)^2 \times 0.05 \left(\cos 33° + \frac{\cos 66°}{4} \right) = 1671 \text{ N}$$

We know that net load on the gudgeon pin,

$$F_P = F_L - F_I = 3520 - 1671 = 1849 \text{ N } \textbf{Ans.}$$

Thrust in the connecting rod

Let φ = Angle of inclination of the connecting rod to the line of stroke.

We know that, $\sin\phi = \frac{\sin\theta}{n} = \frac{\sin 33°}{4} = \frac{0.5446}{4} = 0.1361$

∴ φ = 7.82°

We know that thrust in the connecting rod,

F

F —______ P □ 1849 □

Reaction between the piston and cylinder

We know that reaction between the piston and cylinder,

F_N □ F_P tan φ □ 1849 tan 7.82□□□ 254 N **Ans.**

Engine speed at which the above values will become zero

A little consideration will show that the above values will become zero, if the inertia force on the reciprocating parts (F_I) is equal to the load on the piston (F_L). Let ω_1 be the speed in rad/s, at which $F_I = F_L$.

$$\therefore \quad m_R\,(\omega_1)^2\, r\left(\cos\theta + \frac{\cos 2\theta}{n}\right) = \frac{\pi}{4} D^2 \times p$$

$$1\,(\omega_1)^2 \times 0.05\left(\cos 33^\circ + \frac{\cos 66^\circ}{4}\right) = \frac{\pi}{4} \times (80)^2 \times 0.7 \quad \text{or} \quad 0.047\,(\omega_1)^2 = 3520$$

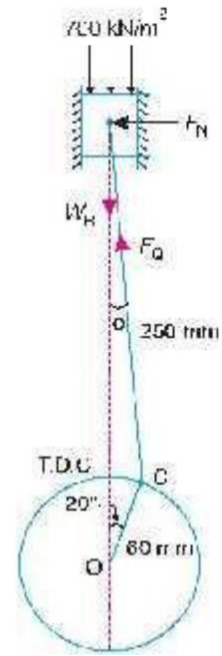
$$\therefore \quad (\omega_1)^2 = 3520 / 0.047 = 74\,894 \text{ or } \omega_1 = 273.6 \text{ rad/s}$$

∴ Corresponding speed in r.p.m.,

$$N_1 = 273.6 \times 60 / 2\pi = 2612 \text{ r.p.m.} \quad \textbf{Ans.}$$

A vertical petrol engine 100 mm diameter and 120 mm stroke has a connecting rod 250 mm long. The mass of the piston is 1.1 kg. The speed is 2000 r.p.m. On the expansion stroke with a crank 20° from top dead centre, the gas pressure is 700 kN/m^2 . Determine:

Net force on the piston, **2.** *Resultant load on the gudgeon pin, Thrust on the cylinder walls, and* **4.** *Speed above which, other things re-maining same, the gudgeon pin load would be reversed in direction.*



Solution. Given: D = 100 mm = 0.1 m ; L = 120 mm = 0.12 m or $r = L/2$ = 0.06 m ; l = 250 mm = 0.25 m ; m_R = 1.1 kg ; N = 2000 r.p.m. or = 2 π × 2000/60 = 209.5 rad/s ; θ = 20°; p = 700 kN/m^2

1. ***Net force on the piston***

The configuration diagram of a vertical engine is shown in Fig. 15.11. We know that force due to gas pressure,

$$F_L = p \times \frac{\pi}{4} \times D^2 = 700 \times \frac{\pi}{4} \times (0.1)^2 = 5.5 \text{ kN}$$
$$= 5500 \text{ N}$$

and ratio of lengths of the connecting rod and crank,

$$n = l/r = 0.25 / 0.06 = 4.17$$

∴ Inertia force on the piston,

We know that for a vertical engine, net force on the piston,

$$F_P = F_L - F_I + W_R = F_L - F_I + m_R\,.g$$
$$= 5500 - 3254 + 1.1 \times 9.81 = 2256.8 \text{ N} \quad \textbf{Ans.}$$

Resultant load on the gudgeon pin

Let φ = Angle of inclination of the connecting rod to the line of stroke. We know that,

$$\sin\varphi = \sin\theta / n = \sin 20°/4.17 = 0.082$$

φ = 4.7°

We know that resultant load on the gudgeon pin,

$$F_Q = \frac{F_P}{\cos\phi} = \frac{2256.8}{\cos 4.7°} = 2265 \text{ N}$$

Thrust on the cylinder walls

We know that thrust on the cylinder walls,

F_N □ F_P tan φ □ 2256.8□ tan 4.□7 □ 185.5 N **Ans.**

Speed, above which, the gudgeon pin load would be reversed in direction

Let N_1 = Required speed, in r.p.m.

The gudgeon pin load *i.e.* F_Q will be reversed in direction, if F_Q becomes negative. This is only possible when F_P is negative. Therefore, for F_P to be negative, F_I must be greater than (F_L + W_R),

$$i.e. \quad m_R (\omega_1)^2 r \left(\cos\theta + \frac{\cos 2\theta}{n}\right) > 5500 + 1.1 \times 9.81$$

$$1.1 \times (\omega_1)^2 \times 0.06 \left(\cos 20° + \frac{\cos 40°}{4.17}\right) > 5510.8$$

$$0.074\,(\omega_1)^2 > 5510.8 \quad \text{or} \quad (\omega_1)^2 > 5510.8/0.074 \quad \text{or} \quad 74\,470$$

$$\text{or} \quad \omega_1 > 273 \text{ rad/s}$$

∴ Corresponding speed in r.p.m.,

$$N_1 > 273 \times 60/2\pi \quad \text{or} \quad 2606 \text{ r.p.m. } \textbf{Ans.}$$

6. *A horizontal steam engine running at 120 r.p.m. has a bore of 250 mm and a stroke of 400 mm. The connecting rod is 0.6 m and mass of the reciprocating parts is 60 kg. When the crank has turned through an angle of 45° from the inne r dead centre, the steam pressure on the cover end side is 550 kN/m 2 and that on the crank end side is 70 kN/m 2 . Considering the diameter of the piston rod equal to 50 mm, determine:*

turning moment on the crank shaft, **2.** *thrust on the bearings, and* **3.** *acceleration of the flywheel, if the power of the engine is 20 kW, mass of the flywheel 60 kg and radius of gyration 0.6 m.*

Solution. Given : N = 120 r.p.m. or ω = 2π × 120/60 = 12.57 rad/s ; D = 250 mm = 0.25 m ;

L = 400 mm = 0.4 m or $r = L/2$ = 0.2 m ; l = 0.6 m ; m_R = 60 kg ; θ = 45° ; d = 50 mm = 0.05 m ; p_1 = 550 kN/m^2 = 550 × 10^3 N/m^2 ; p_2 = 70 kN/m^2 = 70 × 10^3 N/m^2

Turning moment on the crankshaft

First of all, let us find the net load on the piston (F_P).

We know that area of the piston on the cover end side,

$$A_1 = \frac{\pi}{4} \times D^2 = \frac{\pi}{4} \times (0.25)^2 = 0.049 \text{ m}^2$$

and area of piston rod, $$a = \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times (0.05)^2 = 0.00196 \text{ m}^2$$

∴ Net load on the piston,

$$F_L = p_1.A_1 - p_2.A_2 = p_1.A_1 - p_2(A_1 - a)$$
$$= 550 \times 10^3 \times 0.049 - 70 \times 10^3 \,(0.049 - 0.001\,96) = 23\,657 \text{ N}$$

We know that ratio of lengths of the connecting rod and crank,

$$n = l/r = 0.6/0.2 = 3$$

and inertia force on the reciprocating parts,

$$F_I = m_R.\omega^2.r\left(\cos\theta + \frac{\cos 2\theta}{n}\right)$$
$$= 60 \times (12.57)^2 \times 0.2\left(\cos 45° + \frac{\cos 90°}{3}\right) = 1340 \text{ N}$$

∴ Net force on the piston or piston effort,

$$F_P = F_L - F_I = 23\,657 - 1340 = 22\,317 \text{ N} = 22.317 \text{ kN}$$

Let ϕ = Angle of inclination of the connecting rod to the line of stroke.

We know that, $\sin\phi = \sin\theta/n = \sin 45°/3 = 0.2357$

∴ $\phi = 13.6°$

We know that turning moment on the crankshaft,

$$T = \frac{F_P \sin(\theta + \phi)}{\cos\phi} \times r = \frac{22.317 \times \sin(45° + 13.6°)}{\cos 13.6°} \times 0.2 \text{ kN-m}$$
$$= 3.92 \text{ kN-m} = 3920 \text{ N-m}$$ **Ans.**

Thrust on the bearings

We know that thrust on the bearings,

$$F_B = \frac{F_P \cos(\theta + \phi)}{\cos\phi} = \frac{22.317 \times \cos(45° + 13.6°)}{\cos 13.6°} = 11.96 \text{ kN}$$ **Ans.**

3. ***Acceleration of the flywheel***

Given: $P = 20$ kW $= 20 \times 10^3$ W; $m = 60$ kg ; $k = 0.6$ m Let α

= Acceleration of the flywheel in rad/s^2.

We know that mass moment of inertia of the flywheel,

$$I = m.k^2 = 60 \times (0.6)^2 = 21.6 \text{ kg-m}^2$$

∴ Accelerating torque, $T_A = I.\alpha = 21.6\,\alpha$ N-m ...(*i*) and resisting torque, $T_R = \frac{P \times 60}{2\pi N} = \frac{20 \times 10^3 \times 60}{2\pi \times 120} = 1591$ N-m $\quad \left(\because P = \frac{2\pi NT}{60}\right)$

Since the accelerating torque is equal to the difference of torques on the crankshaft or turning moment (T) and the resisting torque (T_R), therefore, accelerating torque,

$$T_A = T - T_R = 3920 - 1591 = 2329 \text{ N-m} \quad ...(ii)$$

From equation (*i*) and (*ii*),

$$\alpha = 2329/21.6 = 107.8 \text{ rad/s}^2$$ **Ans.**

1.10 EQUIVALENT DYNAMICAL SYSTEM

In order to determine the motion of a rigid body, under the action of external forces, it is usually convenient to replace the rigid body by two masses placed at a fixed distance apart, in such a way that,

the sum of their masses is equal to the total mass of the body ;

the centre of gravity of the two masses coincides with that of the body ; and

the sum of mass moment of inertia of the masses about their centre of gravity is equal to the mass moment of inertia of the body.

When these three conditions are satisfied, then it is said to be an ***equivalent dynamical system.*** Consider a rigid body, having its centre of gravity at *G*, as shown in Fig. 15.14.

Let m = Mass of the body,

kG = Radius of gyration about its centre of gravity *G*, *m*1 and *m*2

= Two masses which form a dynamical equivalent system, *l*1 =

Distance of mass *m*1 from *G*, *l*2 = Distance of mass *m*2 from *G*,

$$m_1 + m_2 = m \quad ...(i)$$

$$m_1.l_1 = m_2.l_2 \quad ...(ii)$$

$$m_1 (l_1)^2 + m_2 (l_2)^2 = m (k_G)^2 \quad ...(iii)$$

From equations (*i*) and (*ii*),

$$m_1 = \frac{l_2.m}{l_1 + l_2} \quad ...(iv)$$

and

$$m_2 = \frac{l_1.m}{l_1 + l_2} \quad ...(v)$$

Substituting the value of m_1 and m_2 in equation (*iii*), we have

$$\frac{l_2.m}{l_1 + l_2}(l_1)^2 + \frac{l_1.m}{l_1 + l_2}(l_2)^2 = m (k_G)^2 \quad \text{or} \quad \frac{l_1 l_2 (l_1 + l_2)}{l_1 + l_2} = (k_G)^2$$

$$\therefore \quad l_1 l_2 = (k_G)^2 \quad ...(vi)$$

This equation gives the essential condition of placing the two masses, so that the system becomes dynamical equivalent. The distance of one of the masses (*i.e.* either *l*1 or *l*2) is arbitrary chosen and the other distance is obtained from equation **(*vi*).**

7 *A connecting rod is suspended from a point 25 mm above the centre of small end, and 650 mm above its centre of gravity, its mass being 37.5 kg. When permitted to oscillate, the time period is found to be 1.87 seconds. Find the dynamical equivalent system constituted of two masses, one of which is located at the small end centre.*

Solution. Given : h = 650 mm = 0.65 m ; l_1 = 650 – 25 = 625 mm

= 0.625 m ; m = 37.5 kg ; t_p = 1.87 s

First of all, let us find the radius of gyration (k_G) of the connect- ing rod (considering it is a compound pendulum), about an axis passing through its centre of gravity, G.

We know that for a compound pendulum, time period of oscillation (t_p),

$$1.87 = 2\pi\sqrt{\frac{(k_G)^2 + h^2}{g.h}} \quad \text{or} \quad \frac{1.87}{2\pi} = \sqrt{\frac{(k_G)^2 + (0.65)^2}{9.81 \times 0.65}}$$

Squaring both sides, we have

$$0.0885 = \frac{(k_G)^2 + 0.4225}{6.38}$$

$$(k_G)^2 = 0.0885 \times 6.38 - 0.4225 = 0.1425 \text{ m}^2$$

$$\therefore \quad k_G = 0.377 \text{ m}$$

It is given that one of the masses is located at the small end centre. Let the other mass is located at a distance l_2 from the centre of gravity G, as shown in Fig. 15.19. We know that, for adynamicallyequivalentsystem,

$$l_1.l_2 = (k_G)^2$$

$$\therefore \quad l_2 = \frac{(k_G)^2}{l_1} = \frac{0.1425}{0.625} = 0.228 \text{ m}$$

Let m_1 = Mass placed at the small end centre A, and

m_2 = Mass placed at a distance l_2 from G, *i.e.* at B.

We know that, for a dynamically equivalent system,

$$m_1 = \frac{l_2.m}{l_1 + l_2} = \frac{0.228 \times 37.5}{0.625 + 0.228} = 10 \text{ kg} \quad \textbf{Ans.}$$

and

$$m_2 = \frac{l_1.m}{l_1 + l_2} = \frac{0.625 \times 37.5}{0.625 + 0.228} = 27.5 \text{ kg} \quad \textbf{Ans.}$$

1.11 CORRECTION COUPLE TO BE APPLIED TO MAKE TWO MASS SYSTEM DYNAMICALLY EQUIVALENT

In Art. 15.11, we have discussed the conditions for equivalent dynamical system of two bodies. A little consideration will show that when two masses are placed arbitrarily*, then the conditions (*i*) and (*ii*) as given in Art. 15.11 will only be satisfied. But the condition (*iii*) is not possible to satisfy. This means that the mass moment of inertia of these two masses placed arbitrarily, will differ than that of mass moment of inertia of the rigid body.

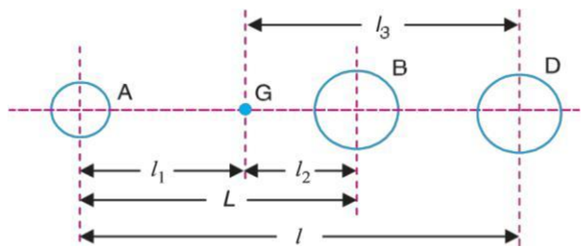


Fig. 15.21. Correction couple to be applied to make the two-mass system dynamically equivalent.

Consider two masses, one at *A* and the other at*D* be placed arbitrarily, as shown in Fig.

15.21. Let l_3 = Distance of mass placed at *D* from *G*, I_1 = New mass

moment of inertia of the two masses; k_1 = New radius of gyration;

α = Angular acceleration of the body;

I = Mass moment of inertia of a dynamically equivalent system; k_G Radius of gyration of a dynamically equivalent system. We know that the torque required to accelerate the body,

$$T = I.\alpha = m (k_G)^2 \alpha \qquad ...(i)$$

Similarly, the torque required to accelerate the two-mass system placed arbitrarily,

$$T_1 = I_1.\alpha = m (k_1)^2 \alpha \qquad ...(ii)$$

$\therefore$ Difference between the torques required to accelerate the two-mass system and the torque required to accelerate the rigid body,

$$T' = T_1 – T = m (k_1)^2 \alpha – m (k_G)^2 \alpha = m [(k_1)^2 – (k_G)^2] \alpha \qquad ...(iii)$$

The difference of the torques *T'* is known as ***correction couple***. This couple must be applied, when the masses are placed arbitrarilyto make the system dynamical equivalent. This, of course, will satisfy the condition (*iii*)

8.A connecting rod of an I.C. engine has a mass of 2 kg and the distance between the centre of gudgeon pin and centre of crank pin is 250 mm. The C.G. falls at a point 100 mm from the gudgeon pin along the line of centres. The radius of gyration about an axis through the C.G. perpendicular to the plane of rotation is 110 mm. Find the equivalent dynamical system if only one of the masses is located at gudgeon pin.

If the connecting rod is replaced by two masses, one at the gudgeon pin and the other at the crank pin and the angular acceleration of the rod is 23 000 rad/s^2 clockwise, determine the correc- tion couple applied to the system to reduce it to a dynamically equivalent system.

Solution. Given :m = 2kg ;l =250 mm =0.25 m ;$l1$ =100 mm= 0.1m ; kG = 110mm= 0.11m ;

 = 23 000 rad/s2

Equivalent dynamical system

It is given that one of the masses is located at the gudgeon pin. Let the other mass be located at a distance $l2$ from the centre of gravity. We know that for an equivalent dynamical system.

$$l_1 l_2 = (k_G)^2 \quad \text{or} \quad l_2 = \frac{(k_G)^2}{l_1} = \frac{(0.11)^2}{0.1} = 0.121 \text{ m}$$

Let m_1 = Mass placed at the gudgeon pin, and

m_2 = Mass placed at a distance l_2 from $C.G.$

We know that
$$m_1 = \frac{l_2 . m}{l_1 + l_2} = \frac{0.121 \times 2}{0.1 + 0.121} = 1.1 \text{ kg} \quad \textbf{Ans.}$$

and
$$m_2 = \frac{l_1 . m}{l_1 + l_2} = \frac{0.1 \times 2}{0.1 + 0.121} = 0.9 \text{ kg} \quad \textbf{Ans.}$$

Correction couple

Since the connecting rod is replaced by two masses located at the two centres (*i.e.* one at the gudgeon pin and the other at the crank pin), therefore,

l = 0.1 m, and $l3$ = l – $l1$ = 0.25 – 0.1 = 0.15 m

Let $k1$ = New radius of gyration.

We know that ($k1$)2= $l1$.$l3$ = 0.1 × 0.15 = 0.015 m2

 Correction couple,

$$T' = m(k_1^2 - k_G^2)\,\alpha = 2\left[0.015 - (0.11)^2\right] 23\,000 = 133.4 \text{ N-m} \quad \textbf{Ans.}$$

1.12 INERTIA FORCES IN A RECIPROCATING ENGINE, CONSIDERING THE WEIGHT OFCONNECTING ROD

In a reciprocating engine, let *OC* be the crank and *PC*, the connecting rod whose centre of gravity lies at *G*. The inertia forces in a reciprocating engine may be obtained graphicallyas discussed below:

First of all, draw the acceleration diagram *OCQN* by Klien’s construction. We know that the acceleration of the piston *P* with respect to *O*,

$aPO = aP = \square^2 \times NO,$

acting in the direction from N to O. Therefore, the inertia force FI of the reciprocating parts will act in the opposite direction as shown in Fig. 15.22.

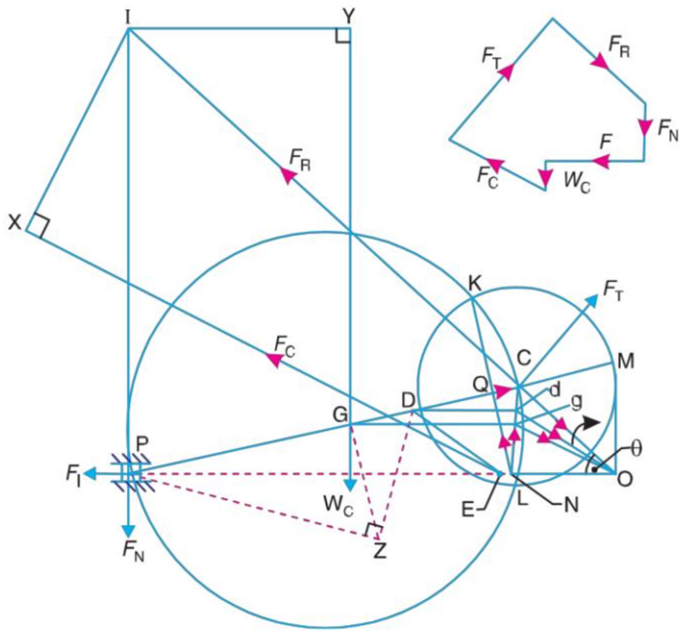


Fig. 15.22. Inertia forces is reciprocating engine, considering the weight of connecting rod.

Replace the connecting rod by dynamically equivalent system of two masses as discussed in Art. 15.12. Let one of the masses be arbitrarily placed at P. To obtain the position of the other mass, draw GZ perpendicular to CP such that $GZ = k$, the radius of gyration of the connecting rod. Join PZ and from Z draw perpendicular to DZ which intersects CP at D. Now, D is the position of the second mass.

Note: The position of the second mass may also be obtained from the equation,

$GP \times GD = k2$

Locate the points G and D on NC which is the acceleration image of the connecting rod. This is done by drawing parallel lines from G and D to the line of stroke PO. Let these parallel lines intersect NC at g and d respectively. Join gO and dO. Therefore, acceleration of G with respect to O, in the direction from g to O,

$= a \quad = 2 \times$ aG O G gO

and acceleration of D with respect to O, in the direction from d to O,

$= a \quad = {}^2 \times$ aD O D dO

4. From D, draw DE parallel to dO which intersects the line of stroke PO at E. Since the accelerating forces on the masses at P and D intersect at E, therefore their resultant must also pass through E. But their resultant is equal to the accelerang force on the rod, so that the line of action of the accelerating force on the rod, is given by a line drawn through E and parallel to gO, in the direc- tion from g to O. The inertia force of the connecting rod F_C therefore acts through E and in the opposite direction as shown in Fig. 15.22. The inertia force of the connecting rod is given by

$$F_C = m_C \times {}^{2} \times gO \qquad \text{...}(\textit{i})$$

where m_C = Mass of the connecting rod.

A little consideration will show that the forces acting on the connecting rod are :

Inertia force of the reciprocating parts (F_I) acting along the line of stroke PO,

The side thrust between the crosshead and the guide bars (F_N) acting at P and right angles to line of stroke PO,

Theweightoftheconnectingrod

($W_C = m_C.g$),

Inertia force of the connecting rod (F_C),

The radial force (F_R) acting through O and parallel to the crank OC,

The force (F_T) acting perpen- dicular to the crank OC.

Now, produce the lines of action of F_R and F_N to intersect at a point I, known as instantaneous centre. From I draw $I\,X$ and $I\,Y$, perpendicular to the lines of action of F_C and W_C. Taking moments about I, we have

$$F_T \times IC = F_I \times IP + F_C \times I\,X + W_C \times I\,Y \quad \text{...}(\textit{ii})$$

The value of F_T may be obtained from this equation and from the force polygon as shown in Fig. 15.22, the forces F_N and F_R may be calculated. We know that, torque exerted on the crankshaft to overcome the inertia of the moving parts = $F_T \times OC$

1.12.1 **Analytical Method for Inertia Torque**

The effect of the inertia of the connecting rod on the crankshaft torque may be obtained as discussed in the following steps:

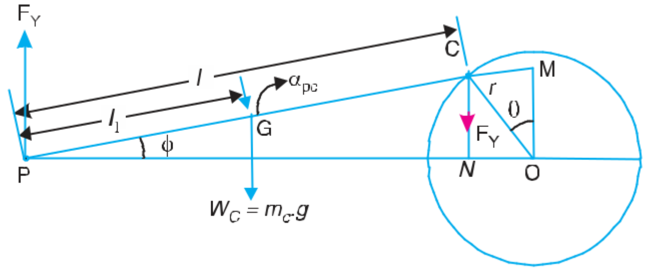


Fig. 15.23. Analytical method for inertia torque.

The mass of the connecting rod (m_C) is divided into two masses. One of the mass is placed at the crosshead pin *P* and the other at the crankpin *C* as shown in Fig. 15.23, so that the centre of gravity of these two masses coincides with the centre of gravity of the rod *G*.

Since the inertia force due to the mass at *C* acts radially outwards along the crank *OC*, therefore the mass at *C* has no effect on the crankshaft torque.

The inertia force of the mass at *P* may be obtained as follows: Let m_C = Mass of the connecting rod, l = Length of the connecting rod, l_1 = Length of the centre of gravity of the connecting rod from *P*.

∴ Mass of the connecting rod at *P*,

$$= \frac{l - l_1}{l} \times m_C$$

The mass of the reciprocating parts (m_R) is also acting at *P*. Therefore,

Total equivalent mass of the reciprocating parts acting at *P*

$$= m_R + \frac{l - l_1}{l} \times m_C$$

∴ Total inertia force of the equivalent mass acting at *P*,

$$F_I = \left(m_R + \frac{l - l_1}{l} \times m_C \right) a_R \quad \text{...(i)}$$

where a_R = Acceleration of the reciprocating parts

$$= \omega^2 r \left(\cos\theta + \frac{\cos 2\theta}{n} \right)$$

$$\therefore \quad F_I = \left[m_R + \frac{l - l_1}{l} \times m_C \right] \omega^2 r \left(\cos\theta + \frac{\cos 2\theta}{n} \right)$$

and corresponding torque exerted on the crank shaft,

$$T_I = F_I \times OM = F_I \, r \left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}} \right) \quad \text{...(ii)}$$

In deriving the equation ***(ii)*** of the torque exerted on the crankshaft, it is assumed that one of the two masses is placed at *C* and the other at *P*. This assumption does not satisfy the condition for kinetically equivalent system of a rigid bar. Hence to compensate for it, a correcting torque is neces- sary whose value is given by

$$T' = m_C \left[(l_1)^2 - (k_G)^2 \right] \alpha_{PC} = m_C \, l_1 \, (l - L) \, \alpha_{PC}$$

where L = Equivalent length of a simple pendulum when swung about an axis through P

$$= \frac{(k_G)^2 + (l_1)^2}{l_1}$$

α_{PC} = Angular acceleration of the connecting rod PC

$$= \frac{-\omega^2 \sin\theta}{n} \quad \text{...(From Art. 15.9)}$$

The correcting torque T' may be applied to the system by two equal and opposite forces *F*Yacting through *P* and *C*. Therefore,

$$F_Y \times PN = T' \quad \text{or} \quad F_Y = T'/PN$$

and corresponding torque on the crankshaft,

$$T_C = F_Y \times NO = \frac{T'}{PN} \times NO \qquad \text{...}(iii)$$

We know that, $\quad NO = OC \cos\theta = r \cos\theta$

and $\quad PN = PC \cos\phi = l \cos\phi$

$$\therefore \quad \frac{NO}{PN} = \frac{r \cos\theta}{l \cos\phi} = \frac{\cos\theta}{n \cos\phi} \qquad \text{...}\left(\because n = \frac{l}{r}\right)$$

$$= \frac{\cos\theta}{n\sqrt{1 - \frac{\sin^2\theta}{n^2}}} = \frac{\cos\theta}{\sqrt{n^2 - \sin^2\theta}} \qquad \left(\because \cos\phi = \sqrt{1 \quad \frac{\sin^2\theta}{n^2}}\right)$$

Since $\sin^2\theta$ is very small as compared to n^2, therefore neglecting $\sin^2\theta$, we have

$$\frac{NO}{PN} = \frac{\cos\theta}{n}$$

Substituting this value in equation *(iii)*, we have

$$T_C = T' \times \frac{\cos\theta}{n} = m_C \times l_1 \, (l - L)\, \alpha_{PC} \times \frac{\cos\theta}{n}$$

$$= -m_C \times l_1 \, (l - L)\, \frac{\omega^2 \sin\theta}{n} \times \frac{\cos\theta}{n} \qquad \text{...}\left(\because \alpha_{PC} = \frac{-\omega^2 \sin\theta}{n}\right)$$

$$= -m_C \times l_1 \, (l - L)\, \frac{\omega^2 \sin 2\theta}{2\,n^2} \qquad \text{...}(\because 2 \sin\theta \cos\theta = \sin 2\theta)$$

5. The equivalent mass of the rod acting at C,

$$m_2 = m_C \times \frac{l_1}{l}$$

$\therefore$ Torque exerted on the crank shaft due to mass m_2,

$$T_W = -m_2 \times g \times NO = -m_C \times g \times \frac{l_1}{l} \times NO = -\, m_C \times g \times \frac{l_1}{l} \times r \cos\theta \qquad \text{...}(\because NO = r\cos\theta)$$

$$= m_C \times g \times \frac{l_1}{n} \times \cos\theta \qquad \text{...}(\because l/r = n)$$

The total torque exerted on the crankshaft due to the inertia of the moving parts is the algebraic sum of T_I, T_C and T_W.

The crank and connecting rod lengths of an engine are 125 mm and 500 mm respectively. The mass of the connecting rod is 60 kg and its centre of gravity is 275 mm from the crosshead pin centre, the radius of gyration about centre of gravity being 150 mm.

If the engine speed is 600 r.p.m. for a crank position of 45° from the inner dead centre, determine, using Klien's or any other construction ***1.*** *the acceleration of the piston;* ***2.*** *the magni- tude, position and direction of inertia force due to the mass of the connecting rod.*

Solution. Given : $r = OC = 125$ mm ; $l = PC = 500$ mm; $m_C = 60$ kg ; $PG = 275$ mm ;

$m_C = 60$ kg ; $PG = 275$ mm ; $k_G = 150$ mm ; $N = 600$ r.p.m. or $\omega = 2\pi \times 600/60 = 62.84$ rad/s

$\theta = 45°$

Acceleration of the piston

Let a_P = Acceleration of the piston.

First of all, draw the configuration diagram *OCP*, as shown in Fig. 15.24, to some suitable scale, such that

$OC = r = 125$ mm ; $PC = l = 500$ mm ; and $\theta = 45°$.

Now, draw the Klien's acceleration diagram *OCQN*, as shown in Fig. 15.24, in the same manner as already discussed. By measurement,

$NO = 90$ mm $= 0.09$ m

$\therefore$ Acceleration of the piston, $a_P = \omega^2 \times NO = (62.84)^2 \times 0.09 = 355.4$ m/s **Ans.**

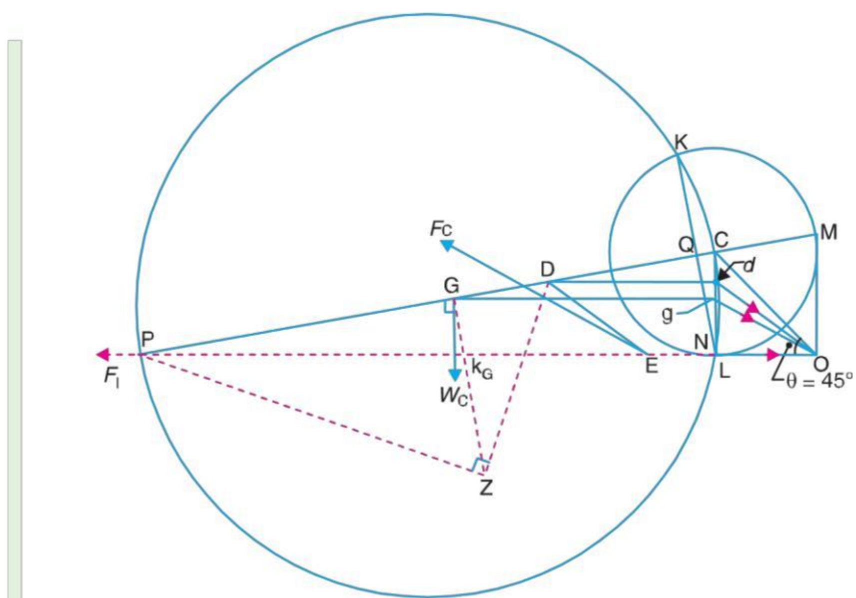


Fig. 15.24

The magnitude, position and direction of inertia force due to the mass of the connecting rod

The magnitude, postition and direction of the inertia force may be obtained as follows:

Replace the connecting rod by dynamical equivalent system of two masses, assuming that one of the masses is placed at *P* and the other mass at *D*. The position of the point *D* is obtained as discussed in Art. 15.12.

Locate the points *G* and *D* on *NC* which is the acceleration image of the connecting rod. Let these points are *g* and *d* on *NC*. Join *gO* and *dO*. By measurement,

$gO = 103$ mm $= 0.103$ m

∴ Acceleration of G, $a_G = \omega^2 \times gO$, acting in the direction from g to O.

From point D, draw DE parallel to dO. Now E is the point through which the inertia force of the connecting rod passes. The magnitude of the inertia force of the connecting rod is given by

$$F_C = m_C \times \omega^2 \times gO = 60 \times (62.84)^2 \times 0.103 = 24\,400 \text{ N} = 24.4$$

kN **Ans. (*iv*)** From point E, draw a line parallel to gO, which shows the position of the inertia force of

the connecting rod and acts in the opposite direction of gO.

The following data refer to a steam engine:

Diameter of piston = 240 mm; stroke = 600 mm ; length of connecting rod = 1.5 m ; mass of reciprocating parts = 300 kg; mass of connecting rod = 250 kg; speed = 125 r.p.m centre of gravity of connecting rod from crank pin = 500 mm ; radius of gyration of the connecting rod about an axis through the centre of gravity = 650 mm.

Determine the magnitude and direction of the torque exerted on the crankshaft when the crank has turned through 30° from inner dead centre.

Solution. Given : D = 240 mm = 0.24 m ; L = 600 mm or $r = L/2$ = 300 mm = 0.3 m ; l = 1.5 m ; m_R = 300 kg ; m_C = 250 kg ; N = 125 r.p.m. or $\omega = 2 \times \pi \times$

125/60 = 13.1 rad/s ; GC = 500 mm = 0.5 m ; k_G = 650 mm = 0.65 m ; θ = 30°

The inertia torque on the crankshaft may be determined by graphical method or analytical method as discussed below:

Graphical method

First of all, draw the configuration diagram OCP, as shown in Fig. 15.25, to some suitable scale, such that

$OC = r$ = 300 mm ; $PC = l$ = 1.5 m ; and angle $POC = \theta$ = 30°.

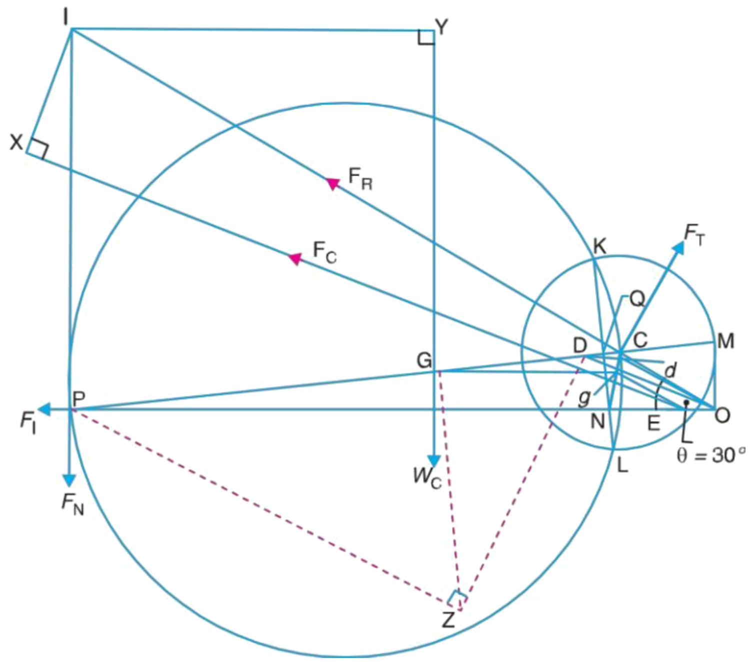


Fig. 15.25

Now draw the Klien's acceleration diagram *OCQN*, as shown in Fig. 15.25, and complete the figure in the similar manner as discussed in Art. 15.14.

By measurement; NO = 0.28 m ; gO = 0.28 m ; IP = 1.03 m ; IX = 0.38 m ; IY = 0.98 m, and IC = 1.7 m.

We know that inertia force of reciprocating parts,

$$F_I = m_R \times \omega^2 \times NO = 300 \times (13.1)^2 \times 0.28 = 14\ 415 \text{ N}$$

and inertia force of connecting rod,

$$F_C = m_C \times \omega^2 \times gO = 250 \times (13.1)^2 \times 0.28 = 12\ 013 \text{ N}$$

Let F_T = Force acting perpendicular to the crank OC.

Taking moments about point I,

$$F_T \times IC = F_I \times IP + W_C \times IY + F_C \times IX$$

$$F_T \times 1.7 - 14\ 415 \times 1.03 + 250 \times 9.81 \times 0.98 + 12013 \times 0.38 - 21816$$

$$\therefore \qquad F_T = 2.816/1.7 = 12\ 833 \text{ N} \qquad \ldots(\because W_C = m_C.g)$$

We know that torque exerted on the crankshaft

$$= F_T \times r = 12\ 833 \times 0.3 = 3850 \text{ N-m}$$ **Ans.**

The connecting rod of an internal combustion engine is 225 mm long and has a mass 1.6 kg. The mass of the piston and gudgeon pin is 2.4 kg and the stroke is 150 mm. The cylinder bore is 112.5 mm. The centre of gravity of the connection rod is 150 mm from the small end. Its radius of gyration about the centre of gravity for oscillations in the plane of *swing of the connect- ing rod is 87.5 mm. Determine the magnitude and direction of the resultant force on the crank pin when the crank is at 40° and the piston is moving away from inner dead centre under an effective gas presure of 1.8 MN/m^2. The engine speed is 1200 r.p.m.*

Solution. Given : $l = PC$ = 225 mm = 0.225 m; m_C = 1.6 kg; m_R = 2.4 kg; L = 150 mm or $r = L/2$ = 75 mm = 0.075 m ; D = 112.5 mm = 0.1125 m ; PG = 150 mm ; k_G = 87.5 mm = 0.0875 m ; θ = 40° ; p = 1.8 MN/m2 = 1.8 × 106 N/m2 ; N = 1200 r.p.m. or ω = 2π × 1200/60 = 125.7 rad/s

First of all, draw the configuration diagram *OCP*, as shown in Fig. 15.27 to some suitable scale, such that *OC* = *r* = 75 mm ; *PC* = *l* = 225 mm ; and θ = 40°.

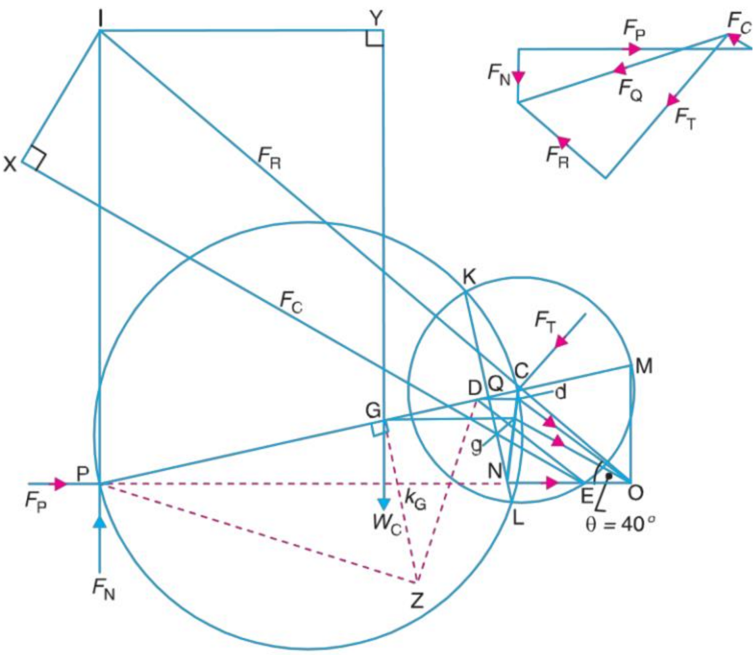


Fig. 15.27

Now, draw the Klien's acceleration diagram*OCQN*. Complete the diagram in the same manner as discussed earlier. By measurement,

NO = 0.0625 m ; *gO* = 0.0685 m ; *IC* = 0.29 m ; *IP* = 0.24 m ; *IY* = 0.148 m ; and *IX* = 0.08 m

We know that force due to gas pressure,

$$F_L = \frac{\pi}{4} \times D^2 \times p = \frac{\pi}{4} \times (0.1125)^2 \times 1.8 \times 10^6 = 17\ 895 \text{ N}$$

Inertia force due to mass of the reciprocating parts,

$$F_I = m_R \times \omega^2 \times NO = 2.4\ (125.7)^2 \times 0.0625 = 2370 \text{ N}$$

∴ Net force on the piston,

$$F_P = F_L - F_I = 17\ 895 - 2370 = 15\ 525 \text{ N}$$

Inertia force due to mass of the connecting rod,

$$F_C = m_C \times \omega^2 \times gO = 1.6 \times (125.7)^2 \times 0.0685 = 1732 \text{ N}$$

We know that lift of the sleeve,

$$h = (r_2 - r_1)\frac{y}{x} = (0.1125 - 0.075)\frac{0.05}{0.1} = 0.018\,75 \text{ m}$$

and stiffness of the spring $s = \frac{S_2 - S}{h} = \frac{705.6 - 426.4}{0.018\,75} = 14\,890 \text{ N/m} = 14.89 \text{ N/mm}$

g)

∴ Initial compression of the spring

$$= \frac{S}{s} = \frac{426.4}{14.89} = 28.6 \text{ mm} \quad \textbf{Ans.}$$

Let us now find the values of *F*N and *F*R in magnitude and direction. Draw the force polygon as shown in Fig. 15.25.

Equilibrium speed corresponding to radius of rotation r = 100 mm = 0.1 m

Let *N* = Equilibrium speed in r.p.m.

By measurement, *F*N = 3550 N; and *F*R = 7550 N.

Since the obliquity of the arms is neglected, therefore the centrifugal force at any instant,

The magnitude and direction of the resultant force on the crank pin is given by *F*Q, which is the resultant of *F*R and *F*T.

$$F_C = F_{C1} + (F_{C2} - F_{C1})\left(\frac{r - r_1}{r_2 - r_1}\right)$$

By measurement, *F*Q = 13 750 N **Ans.**

$$= 106.6 + (176.4 - 106.6)\left(\frac{0.1 - 0.075}{0.1125 - 0.075}\right) = 153 \text{ N}$$

We know that centrifugal force (F_C),

$$153 = m.\omega^2.r = 1\left(\frac{2\pi N}{60}\right)^2 0.1 = 0.0011\,N^2$$

∴ $N^2 = 153 / 0.0011 = 139\,090$ or $N = 373$ r.p.m. **Ans.**