UNIT I - PARTIAL DIFFERENTIAL EQUATIONS

1.3 LAGRANGE'S LINEAR DIFFERENTIAL EQUATION

Lagrange's Linear Differential Equations:

Equations of the form $Pp+Qq=R$ (or) $P\frac{\partial z}{\partial x}+Q\frac{\partial z}{\partial y}=R$

Where P,Q,R are functions of x, y, z or constants.

Procedure :

1. Write the auxiliary equation $\boxed{\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}}$

2. Solve the auxiliary equation by using

   a) Method of grouping

   b) Method of multipliers

a) Method of grouping: In the auxiliary equation, if the variables can be separated in any pair of equations, then we get a solution of the form $u(x,y,z)=c_1$ & $v(x,y,z)=c_2$

$\therefore$ The general solution is $\boxed{\phi(u,v)=0}$

b) Method of Multipliers:

i) Choose any three multipliers l,m,n which may be constants or functions of x, y, z we have

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{ldx+mdy+ndz}{lP+mQ+nR}$$

If it is possible to choose l,m,n such that $lP+mQ+nR=0$ then $ldx+mdy+ndz=0$

Integrating this we get $u(x,y,z)=c_1$

ii) Choose another any three multipliers l',m',n' which may be constants or functions of x,y,z we have

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{l'dx+m'dy+n'dz}{l'P+m'Q+n'R}$$

If it is possible to choose l',m',n' such that $l''P+m'Q+n'R=0$ then $l'dx+m'dy+n'dz=0$

Integrating this we get $v(x,y,z)=c_2$

$\therefore$ The general solution is $\phi(u,v)=0$

1. **Solve** $x(y-z)p+y(z-x)q=z(x-y)$

Solution:

Given $x(y-z)p+y(z-x)q=z(x-y)$

This is of the form $Pp+Qq=R$

Where $P=x(y-z);\ Q=y(z-x);\ R=z(x-y)$

The auxiliary equation be

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)}\quad ----(1)$$

i) Choose the multipliers as (1,1,1)

$$(1)\Rightarrow\frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)}$$

$$=\frac{dx+dy+dz}{xy-xz+yz-xy+xz-yz}=\frac{dx+dy+dz}{0}$$

$\therefore dx+dy+dz=0$

Integrating $\int dx+\int dy+\int dz=0$

	$\boxed{x+y+z=c_1}$ ii) Choose the multipliers as $\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$ $(1)\Rightarrow \frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)}$ $=\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y-z+z-x+x-y}=\frac{dx+dy+dz}{0}$ $\therefore \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$ Integrating $\int\frac{dx}{x}+\int\frac{dy}{y}+\int\frac{dz}{z}=0$ $\log x+\log y+\log z=\log c_2$ $\log xyz=\log c_2 \quad \Rightarrow \boxed{xyz=c_2}$ $\therefore$ The general solution is $\boxed{\phi\left(x+y+z,xyz\right)=0}$
2.	**Solve** $x(z^2-y^2)p+y(x^2-z^2)q=z(y^2-x^2)$ **Solution:** Given $x(z^2-y^2)p+y(x^2-z^2)q=z(y^2-x^2)$ This is of the form $Pp+Qq=R$ Where $P=x(z^2-y^2);\ Q=y(x^2-z^2);\ R=z(y^2-x^2)$ The auxiliary equation be $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ $\frac{dx}{x(z^2-y^2)}=\frac{dy}{y(x^2-z^2)}=\frac{dz}{z(y^2-x^2)}\ ----(1)$

i) Choose the multipliers as (x,y,z)

$$(1) \Rightarrow \frac{xdx}{x^2(z^2-y^2)} = \frac{ydy}{y^2(x^2-z^2)} = \frac{zdz}{z^2(y^2-x^2)}$$

$$= \frac{xdx+ydy+zdz}{x^2z^2-x^2y^2+x^2y^2-y^2z^2+y^2z^2-x^2z^2} = \frac{xdx+ydy+zdz}{0}$$

$$\therefore\ xdx+ydy+zdz=0$$

Integrating $\int xdx+\int ydy+\int zdz=0$

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=\frac{c_1^2}{2}$$

$$\boxed{x^2+y^2+z^2=c_1^2}$$

ii) Choose the multipliers as $\left(\frac{1}{x},\frac{1}{y},\frac{1}{z}\right)$

$$(1) \Rightarrow \frac{\frac{dx}{x}}{\frac{x}{x}(z^2-y^2)} = \frac{\frac{dy}{y}}{\frac{y}{y}(x^2-z^2)} = \frac{\frac{dz}{z}}{\frac{z}{z}(y^2-x^2)}$$

$$= \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{z^2-y^2+x^2-z^2+y^2-x^2} = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

$$\therefore \frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

Integrating $\int\frac{dx}{x}+\int\frac{dy}{y}+\int\frac{dz}{z}=0$

$$\log x+\log y+\log z=\log c_2$$

$$\log xyz=\log c_2 \qquad \Rightarrow \boxed{xyz=c_2}$$

$\therefore$ The general solution is $\boxed{\phi\left(x^2+y^2+z^2, xyz\right)=0}$

3.	**Solve** $(x^2-y^2-z^2)p+2xyq=2zx$ **Solution:** Given $(x^2-y^2-z^2)p+2xyq=2zx$ This is of the form $Pp+Qq=R$ Where $P=(x^2-y^2-z^2);\ Q=2xy;\ R=2zx$ The auxiliary equation be $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ $\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2zx}\ ----(1)$ i) by method of grouping, from last two ratios $(1)\Rightarrow\frac{dy}{2xy}=\frac{dz}{2zx}$ $\frac{dy}{y}=\frac{dz}{z}$ Integrating $\int\frac{dy}{y}=\int\frac{dz}{z}$ $\log y=\log z+\log c_1$ $\log y-\log z=\log c_1$ $\log\frac{y}{z}=\log c_1$ $\boxed{\frac{y}{z}=c_1}$ ii) Choose the multipliers as (x,y,z)

	$(1) \Rightarrow \frac{xdx}{x\left(x^2-y^2-z^2\right)}=\frac{ydy}{y\left(2xy\right)}=\frac{zdz}{z\left(2zx\right)}$ $=\frac{xdx+ydy+zdz}{x^3-xy^2-xz^2+2xy^2+2xz^2}=\frac{xdx+ydy+zdz}{x^3+xy^2+xz^2}$ $=\frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}$ From 3^{rd} and last ratio $\frac{dz}{\left(2zx\right)}=\frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}$ $\frac{dz}{z}=\frac{2xdx+2ydy+2zdz}{(x^2+y^2+z^2)}$ Integrating $\int\frac{dz}{z}=\int\frac{2xdx+2ydy+2zdz}{(x^2+y^2+z^2)}$ $\log z=\log\left(x^2+y^2+z^2\right)$ $\quad\because\int\frac{f'(x)}{f(x)}dx=\log\left[f(x)\right]$ $\log z=\log\left(x^2+y^2+z^2\right)+\log c_2$ $\log z-\log\left(x^2+y^2+z^2\right)=\log c_2$ $\log\frac{z}{x^2+y^2+z^2}=\log c_2$ $\boxed{\frac{z}{x^2+y^2+z^2}=c_2}$ The general solution is $\boxed{\phi\left(\frac{y}{z},\frac{z}{x^2+y^2+z^2}\right)=0}$
4.	**Solve** $(3z-4y)p+(4x-2z)q=2y-3x$ **Hint:**

	The multipliers are (x, y, z) & $(2,3,4)$ The general solution is $\phi\left(x^2+y^2+z^2, 2x+3y+4z\right)=0$
5.	**Solve** $(y-xz)p+(yz-x)q=(x+y)(x-y)$ **Hint:** The multipliers are (x, y, z) & $(y, x, 1)$ The general solution is $\phi\left(x^2+y^2+z^2, xy+z\right)=0$
6.	**Solve** $x(y^2+z)p+y(x^2+z)q=z(x^2-y^2)$ **Hint:** The multipliers are $\left(\frac{1}{x}, \frac{-1}{y}, \frac{1}{z}\right)$ & $(x, -y, -1)$ The general solution is $\phi\left(\frac{xz}{y}, x^2-y^2-2z\right)=0$