# Chapter-1 Physical Properties of Fluids

## 1.0 INTRODUCTION

The flow of ideal non-viscous fluids was extensively studied and mathematical theories were developed during the last century. The field of study was called as 'Hydrodynamics'. However the results of mathematical analysis could not be applied directly to the flow of real fluids. Experiments with water flow resulted in the formulation of empirical equations applicable to engineering designs. The field was called Hydraulics. Due to the development of industries there arose a need for the study of fluids other than water. Theories like boundary layer theory were developed which could be applied to all types of real fluids, under various conditions of flow. The combination of experiments, the mathematical analysis of hydrodynamics and the new theories is known as 'Fluid Mechanics'. **Fluid Mechanics encompasses the study of all types of fluids under static, kinematic and dynamic conditions.**

The study of properties of fluids is basic for the understanding of flow or static condition of fluids. The important properties are **density, viscosity, surface tension, bulk modulus and vapour pressure.** Viscosity causes resistance to flow. Surface tension leads to capillary effects. Bulk modulus is involved in the propagation of disturbances like sound waves in fluids. Vapour pressure can cause flow disturbances due to evaporation at locations of low pressure. It plays an important role in cavitation studies in fluid machinery.

In this chapter various properties of fluids are discussed in detail, with stress on their effect on flow. Fairly elaborate treatment is attempted due to their importance in engineering applications. The basic laws used in the discussions are :

(*i*) Newton's laws of motion,

(*ii*) Laws of conservation of mass and energy,

(*iii*) Laws of Thermodynamics, and

(*iv*) Newton's law of viscosity.

**A fluid is defined as a material which will continue to deform with the application of shear force however small the force may be.**

## 1.1 THREE PHASES OF MATTER

Generally matter exists in three phases namely (*i*) Solid (*ii*) Liquid and (*iii*) Gas (includes vapour). The last two together are also called by the common term **fluids**.

In solids atoms/molecules are closely spaced and the attractive (cohesive) forces between atoms/molecules is high. The shape is maintained by the cohesive forces binding the atoms. When an external force is applied on a solid component, slight rearrangement in atomic positions balances the force. Depending upon the nature of force the solid may elongate or shorten or bend. When the applied force is removed the atoms move back to the original position and the former shape is regained. Only when the forces exceed a certain value (yield), a small deformation called plastic deformation will be retained as the atoms are unable to move to their original positions. When the force exceeds a still higher value (ultimate), the cohesive forces are not adequate to resist the applied force and the component will break.

In liquids the inter molecular distances are longer and the cohesive forces are of smaller in magnitude. The molecules are not bound rigidly as in solids and can move randomly. However, the cohesive forces are large enough to hold the molecules together below a free surface that forms in the container. Liquids will continue to deform when a shear or tangential force is applied. The deformation continues as long as the force exists. In fluids the **rate of deformation** controls the force (not deformation as in solids). More popularly it is stated that a fluid (liquid) cannot withstand applied shear force and will continue to deform. When at rest liquids will assume the shape of the container forming a free surface at the top.

In gases the distance between molecules is much larger compared to atomic dimensions and the cohesive force between atoms/molecules is low. So gas molecules move freely and fill the full volume of the container. If the container is open the molecules will diffuse to the outside. Gases also cannot withstand shear. The **rate of deformation** is proportional to the applied force as in the case of liquids.

Liquids and gases together are classified as fluids. Vapour is gaseous state near the evaporation temperature. The state in which a material exists depends on the pressure and temperature. For example, steel at atmospheric temperature exists in the solid state. At higher temperatures it can be liquefied. At still higher temperatures it will exist as a vapour.

A fourth state of matter is its existence as charged particles or ions known as plasma. This is encountered in MHD power generation. This phase is not considered in the text.

## 1.2 COMPRESSIBLE AND INCOMPRESSIBLE FLUIDS

**If the density of a fluid varies significantly due to moderate changes in pressure or temperature, then the fluid is called compressible fluid**. Generally gases and vapours under normal conditions can be classified as compressible fluids. In these phases the distance between atoms or molecules is large and cohesive forces are small. So increase in pressure or temperature will change the density by a significant value.

**If the change in density of a fluid is small due to changes in temperature and or pressure, then the fluid is called incompressible fluid.** All liquids are classified under this category.

When the change in pressure and temperature is small, gases and vapours are treated as incompressible fluids. For certain applications like propagation of pressure disturbances, liquids should be considered as compressible.

In this chapter some of the properties relevant to fluid mechanics are discussed with a view to bring out their influence on the design and operation of fluid machinery and equipments.

## 1.3 DIMENSIONS AND UNITS

It is necessary to distinguish clearly between the terms “Units” and “Dimensions”. The word “dimension” is used to describe basic concepts like mass, length, time, temperature and force. “Large mass, long distance, high temperature” does not mean much in terms of visualising the quantity. Dimension merely describes the concept and does not provide any method for the quantitative expression of the same. Units are the means of expressing the value of the dimension quantitatively or numerically The term “second” for example is used to quantify time. “Ten seconds elapsed between starting and ending of an act” is the way of expressing the elapsed time in numerical form. The value of dimension should be expressed in terms of units before any quantitative assessment can be made.

There are three widely used systems of units in the world. These are (1) British or English system (it is not in official use now in Briton) (2) Metric system and (3) SI system (System International d’Unites or International System of Units). India has passed through the first two systems in that order and has now adopted the SI system of units.

The basic units required in Fluid Mechanics are for mass, length, time and temperature. These are kilogram (**kg**), metre (**m**), second (**s**) and kelvin (**K**). The unit of force is defined using Newton’s second law of motion which states that applied force is proportional to the time rate of change of momentum of the body on which the force acts.

For a given mass **m**, subjected to the action of a force **F**, resulting in an acceleration **a**, Newton’s law can be written in the form

$$\mathbf{F = (1/g_o)\ m\ a} \tag{1.3.1}$$

where $g_o$ is a dimensional constant whose numerical value and units depend on those selected for **force, F, mass, m, and acceleration, a**. The unit of force is newton (N) in the SI system.

One newton is defined as the force which acting on a mass of one kilogram will produce an acceleration of 1 m/s$^2$. This leads to the relation

$$\mathbf{1\ N = (1/g_o) \times 1\ kg \times 1\ m/s^2} \tag{1.3.2}$$

Hence $$\mathbf{g_o = 1\ kg\ m/N\ s^2} \tag{1.3.3}$$

The numerical value of $g_o$ is unity (1) in the SI system and this is found advantageous in numerical calculations. However this constant should necessarily be used to obtain dimensional homogeneity in equations.

In metric system the unit of force is $kg_f$ defined as the force acted on one kg mass by standard gravitational acceleration taken as 9.81 m/s$^2$. The value of $g_o$ is 9.81 kg m/$kg_f s^2$.

In the English system the unit of force is $lb_f$ defined as the force on one lb mass due to standard gravitational acceleration of 32.2 ft/s$^2$.

The value of $g_o$ is 32.2 ft lb/$lb_f s^2$.

Some of the units used in this text are listed in the table below:

| Quantity | Unit symbol | Derived units |
|---|---|---|
| mass | kg | ton (tonne) = 1000 kg |
| time | s | min (60s), hr (3600s) |
| length | m | mm, cm, km |
| temperature | K, (273 + °C) | °C |
| force | N (newton) | kN, MN ($10^6$ N) |
| energy, work, heat | Nm, J | kJ, MJ, kNm |
| power | *W* = (Nm/s, J/s) | kW, MW |
| pressure | N/$m^2$, (pascal, pa) | kPa, MPa, bar ($10^5$Pa) |

Conversion constants between the metric and SI system of units are tabulated elsewhere in the text.

## 1.4 CONTINUUM

As gas molecules are far apart from each other and as there is empty space between molecules doubt arises as to whether a gas volume can be considered as a continuous matter like a solid for situations similar to application of forces.

Under normal pressure and temperature levels, gases are considered as a continuum (*i.e.,* as if no empty spaces exist between atoms). The test for continuum is to measure properties like density by sampling at different locations and also reducing the sampling volume to low levels. If the property is constant irrespective of the location and size of sample volume, then the gas body can be considered as a continuum for purposes of mechanics (application of force, consideration of acceleration, velocity etc.) and for the gas volume to be considered as a single body or entity. This is a very important test for the application of all laws of mechanics to a gas volume as a whole. When the pressure is extremely low, and when there are only few molecules in a cubic metre of volume, then the laws of mechanics should be applied to the molecules as entities and not to the gas body as a whole. In this text, only systems satisfying continuum requirements are discussed.

## 1.5 DEFINITION OF SOME COMMON TERMINOLOGY

***Density (mass density)*****:** The mass per unit volume is defined as density. The unit used is kg/$m^3$. The measurement is simple in the case of solids and liquids. In the case of gases and vapours it is rather involved. The symbol used is $\rho$. The characteristic equation for gases provides a means to estimate the density from the measurement of pressure, temperature and volume.

***Specific Volume*****:** The volume occupied by unit mass is called the specific volume of the material. The symbol used is $v$, the unit being $m^3$/kg. Specific volume is the reciprocal of density.

In the case of solids and liquids, the change in density or specific volume with changes in pressure and temperature is rather small, whereas in the case of gases and vapours, density will change significantly due to changes in pressure and/or temperature.

***Weight Density or Specific Weight*:** The force due to gravity on the mass in unit volume is defined as Weight Density or Specific Weight. The unit used is $N/m^3$. The symbol used is $\gamma$. At a location where $g$ is the local acceleration due to gravity,

$$\textbf{Specific weight, } \gamma = g\,\rho \qquad (1.5.1)$$

In the above equation direct substitution of dimensions will show apparent non-homogeneity as the dimensions on the LHS and RHS will not be the same. On the LHS the dimension will be $N/m^3$ but on the RHS it is $kg/m^2\,s^2$. The use of $g_o$ will clear this anomaly. As seen in section 1.1, $g_o$ = 1 kg m/N $s^2$. The RHS of the equation 1.3.1 when divided by $g_o$ will lead to perfect dimensional homogeneity. The equation should preferably be written as,

$$\textbf{Specific weight, } \gamma = (g/g_o)\,\rho \qquad (1.5.2)$$

Since newton (N) is defined as the force required to accelerate 1 kg of mass by $1/s^2$, it can also be expressed as kg.m/$s^2$. Density can also be expressed as $Ns^2/m^4$ (as kg = $Ns^2$/m). Beam balances compare the mass while spring balances compare the weights. The mass is the same (invariant) irrespective of location but the weight will vary according to the local gravitational constant. Density will be invariant while specific weight will vary with variations in gravitational acceleration.

***Specific Gravity or Relative Density*:** The ratio of the density of the fluid to the density of water—usually 1000 $kg/m^3$ at a standard condition—is defined as Specific Gravity or Relative Density $\delta$ of fluids. This is a ratio and hence no dimension or unit is involved.

**Example 1.1.** *The weight of an object measured on ground level where $g_e$ = 9.81 $m/s^2$ is 35,000 N. Calculate its weight at the following locations (i) Moon, $g_m$ = 1.62 $m/s^2$ (ii) Sun, $g_s$ = 274.68 $m/s^2$ (iii) Mercury, $g_{me}$ = 3.53 $m/s^2$ (iv) Jupiter, $g_j$ = 26.0 $m/s^2$ (v) Saturn, $g_{sa}$ = 11.2 $m/s^2$ and (vi) Venus, $g_v$ = 8.54 $m/s^2$.*

Mass of the object, $m_e$ = weight × $(g_o/g)$ = 35,000 × (1/9.81) = 3567.8 kg

Weight of the object on a planet, $p = m_e \times (g_p/g_o)$ where $m_e$ is the mass on earth, $g_p$ is gravity on the planet and $g_o$ has the usual meaning, force conversion constant.

Hence the weight of the given object on,

| | | | | |
|---|---|---|---|---|
| (*i*) | Moon | = | 3567.8 × 1.62 | = 5,780 N |
| (*ii*) | Sun | = | 3567.8 × 274.68 | = 9,80,000 N |
| (*iii*) | Mercury | = | 3567.8 × 3.53 | = 12,594 N |
| (*iv*) | Jupiter | = | 3567.8 × 26.0 | = 92,762 N |
| (*v*) | Saturn | = | 3567.8 × 11.2 | = 39,959 N |
| (*vi*) | Venus | = | 3567.8 × 8.54 | = 30,469 N |

Note that the mass is constant whereas the weight varies directly with the gravitational constant. Also note that the ratio of weights will be the same as the ratio of gravity values.

## 1.6 VAPOUR AND GAS

When a liquid is heated under a constant pressure, first its temperature rises to the boiling point (defined as saturation temperature). Then the liquid begins to change its phase to the gaseous condition, with molecules escaping from the surface due to higher thermal energy level. **When the gas phase is in contact with the liquid or its temperature is near the saturation condition it is termed as vapour.**

Vapour is in gaseous condition but it does not follow the gas laws. Its specific heats will vary significantly. Moderate changes in temperature may change its phase to the liquid state.

When the temperature is well above the saturation temperature, vapour begins to behave as a gas. It will also obey the characteristic equation for gases. Then the specific heat will be nearly constant.

## 1.7 CHARACTERISTIC EQUATION FOR GASES

The characteristic equation for gases can be derived from Boyle's law and Charles' law. Boyle's law states that at constant temperature the volume of a gas body will vary inversely with pressure. Charles' law states that at constant pressure, the temperature will vary inversely with volume. Combining these two, the characteristic equation for a system containing **m** kg of a gas can be obtained as

$$\mathbf{PV = mRT} \tag{1.7.1}$$

This equation when applied to a given system leads to the relation 1.7.2 applicable for all equilibrium conditions irrespective of the process between the states.

$$\mathbf{(P_1V_1/T_1) = (P_2V_2/T_2) = (P_3\ V_3/T_3) = (PV/T) = Constant} \tag{1.7.2}$$

In the SI system, the units to be used in the equation are Pressure, $P \rightarrow$ N/m$^2$, volume, $V \rightarrow$ m$^3$, mass, m $\rightarrow$ kg, temperature, $T \rightarrow$ K and gas constant, $R \rightarrow$ Nm/kgK or J/kgK (Note: K = (273 + °C), J = Nm).

This equation defines the equilibrium state for any gas body. For a specified gas body with mass **m,** if two properties like **P, V** are specified then the third property **T** is automatically specified by this equation. The equation can also be written as,

$$\mathbf{Pv = RT} \tag{1.7.3}$$

where $v = V$/m or specific volume. **The value for R for air is 287 J/kgK.**

Application of Avagadro's hypothesis leads to the definition of a new volume measure called molal volume. This is the volume occupied by the molecular mass of any gas at standard temperature and pressure. This volume as per the above hypothesis will be the same for all gases at any given temperature and pressure. Denoting this volume as $V_m$ and the pressure as $P$ and the temperature as $T$,

**For a gas a,** $$\mathbf{PV_m = M_a\ R_a\ T} \tag{1.7.4}$$

**For a gas b,** $$\mathbf{PV_m = M_b\ R_b\ T} \tag{1.7.5}$$

As P, T and $V_m$ are the same in both cases.

$$\mathbf{M_aR_a = M_bR_b = M \times R = Constant} \tag{1.7.6}$$

The product **M × R** is called **Universal gas constant** and is denoted by the symbol $\mathfrak{R}$. Its numerical value in SI system is **8314 J/kg mole K**. For any gas the value of gas constant $R$ is obtained by dividing universal gas constant by the molecular mass in kg of that gas. The gas constant R for any gas (in the SI system, J/kg K) can be calculated using,

$$\mathbf{R = 8314/M} \tag{1.7.7}$$

The characteristic equation for gases can be applied for all gases with slight approximations, and for practical calculations this equation is used in all cases.

**Example 1.2.** *A balloon is filled with 6 kg of hydrogen at 2 bar and 20°C.* ***What will be the diameter of the balloon*** *when it reaches an altitude where the pressure and temperature are 0.2 bar and –60° C. Assume that the pressure and temperature inside are the same as that at the outside at this altitude.*

The characteristic equation for gases $PV = mRT$ is used to calculate the initial volume,

$V_1 = [(m\,RT_1)/P_1]$, For hydrogen, molecular mass = 2, and so

$R_H = 8314/2 = 4157$ J/kgK, $\therefore$ $V_1 = 6 \times 4157 \times (273 + 20)/2 \times 10^5 = 36.54$ m$^3$

Using the general gas equation the volume after the balloon has reached the altitude, $V_2$ is calculated. $[(P_1V_1)/T_1] = [(P_2V_2)/T_2]$

$[(2 \times 10^5 \times 36.54)/(273+20)] = [(0.2) \times 10^5 \times V_2)/(273 - 60)]$ solving,

$V_2 =$ **265.63 m$^3$**, Considering the shape of the balloon as a sphere of radius r,

Volume = $(4/3)\,\pi\, r^3 = 265.63$ m$^3$, solving

**Radius, r = 3.99 m** and **diameter of the balloon = 7.98 m**

(The pressure inside the balloon should be slightly higher to overcome the stress in the wall material)

## 1.8 VISCOSITY

A fluid is defined as a material which will continue to deform with the application of a shear force. However, different fluids deform at different rates when the same shear stress (force/area) is applied.

**Viscosity is that property of a real fluid by virtue of which it offers resistance to shear force.** Referring to Fig. 1.8.1, it may be noted that a force is required to move one layer of fluid over another.

For a given fluid the force required varies directly as the rate of deformation. As the rate of deformation increases the force required also increases. This is shown in Fig. 1.8.1 (*i*).

The force required to cause the same rate of movement depends on the nature of the fluid. The resistance offered for the same rate of deformation varies directly as the viscosity of the fluid. As viscosity increases the force required to cause the same rate of deformation increases. This is shown in Fig. 1.8.1 (*ii*).

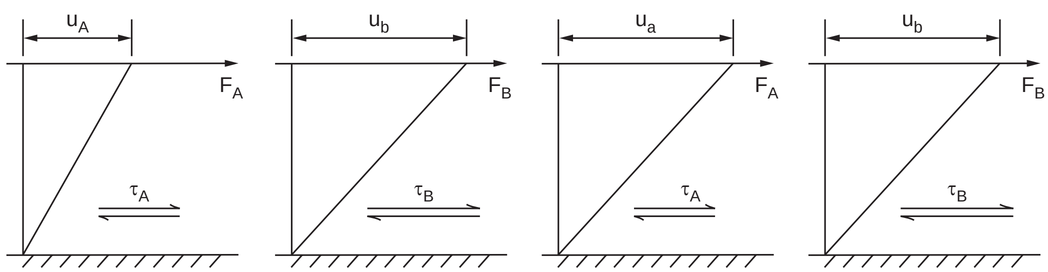
Newton's law of viscosity states that the shear force to be applied for a deformation rate of ($du/dy$) over an area A is given by,

$$\mathbf{F = \mu\, A\, (du/dy)} \quad (1.8.1)$$

or

$$\mathbf{(F/A) = \tau = \mu\, (du/dy) = \mu\, (u/y)} \quad (1.8.2)$$

where $F$ is the applied force in N, $A$ is area in m$^2$, $du/dy$ is the velocity gradient (or rate of deformation), 1/s, perpendicular to flow direction, here assumed linear, and $\mu$ is the proportionality constant defined as the **dynamic or absolute viscosity** of the fluid.




$u_b > u_a$ , $F_b > F_a$ $\mu_a = \mu_b$
(*i*) same fluid

$u_a = u_b$ , $\mu_a < \mu_b$ , $F_b > F_a$
(*ii*) same velocity

**Figure 1.8.1** Concept of viscosity

The dimensions for dynamic viscosity $\mu$ can be obtained from the definition as Ns/m$^2$ or kg/ms. The first dimension set is more advantageously used in engineering problems. However, if the dimension of N is substituted, then the second dimension set, more popularly used by scientists can be obtained. The numerical value in both cases will be the same.

$$N = kg\ m/s^2\ ;\ \mu = (kg\ m/s^2)\ (s/m^2) = kg/ms$$

The popular unit for viscosity is Poise named in honour of Poiseuille.

$$\textbf{Poise = 0.1 Ns/m}^2 \qquad (1.8.3)$$

Centipoise (cP) is also used more frequently as,

$$\textbf{cP = 0.001 Ns/m}^2 \qquad (1.8.3a)$$

For water the viscosity at 20°C is nearly 1 cP. The ratio of dynamic viscosity to the density is defined as kinematic viscosity, $\nu$, having a dimension of m$^2$/s. Later it will be seen to relate to momentum transfer. Because of this kinematic viscosity is also called momentum diffusivity. The popular unit used is stokes (in honour of the scientist Stokes). Centistoke is also often used.

$$\textbf{1 stoke = 1 cm}^2\textbf{/s = 10}^{-4}\ \textbf{m}^2\textbf{/s} \qquad (1.8.3b)$$

Of all the fluid properties, viscosity plays a very important role in fluid flow problems. The velocity distribution in flow, the flow resistance etc. are directly controlled by viscosity. In the study of fluid statics (*i.e.,* when fluid is at rest), viscosity and shear force are not generally involved. In this chapter problems are worked assuming linear variation of velocity in the fluid filling the clearance space between surfaces with relative movement.

**Example 1.3.** *The space between two large inclined parallel planes is 6mm and is filled with a fluid. The planes are inclined at 30° to the horizontal. A small thin square plate of 100 mm side slides freely down parallel and midway between the inclined planes with a constant velocity of 3 m/s due to its weight of 2N.* ***Determine the viscosity of the fluid.***

The vertical force of 2 N due to the weight of the plate can be resolved along and perpendicular to the inclined plane. The force along the inclined plane is equal to the drag force on both sides of the plane due to the viscosity of the oil.

Force due to the weight of the sliding plane along the direction of motion

$$= 2 \sin 30 = 1N$$

Viscous force, F = (A × 2) × μ × ($du/dy$) (both sides of plate). Substituting the values,

$$1 = \mu \times [(0.1 \times 0.1 \times 2)] \times [(3 - 0)/6/(2 \times 1000)\}]$$

Solving for viscosity, **μ = 0.05 Ns/m$^2$ or 0.5 Poise**

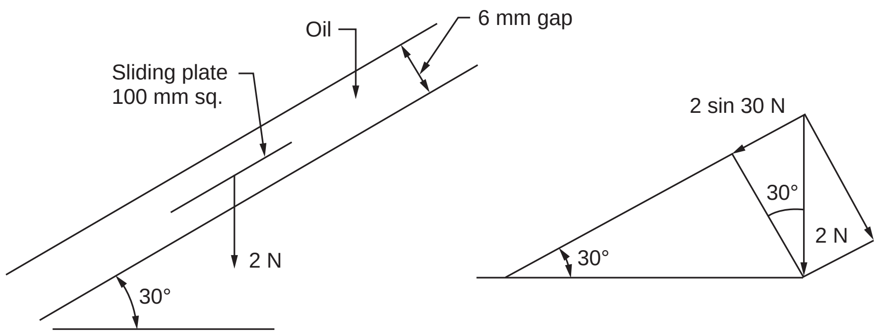



**Figure Ex. 1.3**

**Example 1.4.** *The velocity of the fluid filling a hollow cylinder of radius 0.1 m varies as u = 10 [1 – (r/0.1)$^2$] m/s along the radius r. The viscosity of the fluid is 0.018 Ns/m$^2$. For 2 m length of the cylinder,* ***determine the shear stress and shear force over cylindrical layers of fluid*** *at r = 0 (centre line), 0.02, 0.04, 0.06 0.08 and 0.1 m (wall surface.)*

Shear stress = μ ($du/dy$) or μ ($du/dr$), $u = 10\ [1 - (r/0.1)^2]$ m/s

∴ $du/dr = 10\ (-2r/0.1^2) = -2000\ r$

The – ve sign indicates that the force acts in a direction opposite to the direction of velocity, $u$. Shear stress = 0.018 × 2000 $r$ = 36 rN/m$^2$

Shear force over 2 m length = shear stress × area over 2$m$

$$= 36r \times 2\pi rL = 72\ \pi r^2 \times 2 = 144\ \pi r^2$$

The calculated values are tabulated below:

| Radius, m | Shear stress, N/m$^2$ | Shear force, N | Velocity, m/s |
|---|---|---|---|
| 0.00 | 0.00 | 0.00 | 0.00 |
| 0.02 | 0.72 | 0.18 | 9.60 |
| 0.04 | 1.44 | 0.72 | 8.40 |
| 0.06 | 2.16 | 1.63 | 6.40 |
| 0.08 | 2.88 | 2.90 | 3.60 |
| 0.10 | 3.60 | 4.52 | 0.00 |

**Example 1.5.** *The 8 mm gap between two large vertical parallel plane surfaces is filled with a liquid of dynamic viscosity 2 × 10$^{-2}$ Ns/m$^2$. A thin sheet of 1 mm thickness and 150 mm × 150 mm size, when dropped vertically between the two plates attains a steady velocity of 4 m/s.* ***Determine weight of the plate.*** *Assume that the plate moves centrally.*

$F = \tau\ (A \times 2) = \mu \times (du/dy)\ (A \times 2)$ = weight of the plate.

Substituting the values, $dy$ = [(8 – 1)/(2 × 1000)] m and $du$ = 4 m/s

$F = 2 \times 10^{-2}\ [4/\{(8 - 1)/(2 \times 1000)\}]\ [0.15 \times 0.15 \times 2]$ = **1.02 N (weight of the plate)**

**Example 1.6.** ***Determine the resistance offered to the downward sliding*** *of a shaft of 400 mm dia and 0.1 m length by the oil film between the shaft and a bearing of ID 402 mm. The kinematic viscosity is $2.4 \times 10^{-4}$ m²/s and density is 900 kg/m³. The shaft is to move centrally and axially at a constant velocity of 0.1 m/s.*

Force, $F$ opposing the movement of the shaft = shear stress × area

$$F = \mu\,(du/dy)\,(\pi \times D \times L)$$

$$\mu = 2.4 \times 10^{-4} \times 900 \text{ Ns/m}^2, \; du = 0.1 \text{ m/s}, \; L = 0.1 \text{ m}, \; D = 0.4 \text{ m}$$

$$dy = (402 - 400)/(2 \times 1000)\text{m, Substituting,}$$

$$F = 2.4 \times 10^{-4} \times 900 \times \{(0.1 - 0)/[(402 - 400)/(2 \times 1000)]\}\,(\pi \times 0.4 \times 0.1) = \mathbf{2714\ N}$$

## 1.8.1 Newtonian and Non Newtonian Fluids

An ideal fluid has zero viscosity. Shear force is not involved in its deformation. An ideal fluid has to be also incompressible. Shear stress is zero irrespective of the value of $du/dy$. Bernoulli equation can be used to analyse the flow.

Real fluids having viscosity are divided into two groups namely Newtonian and non Newtonian fluids. In Newtonian fluids a linear relationship exists between the magnitude of the applied shear stress and the resulting rate of deformation. It means that the proportionality parameter (in equation 1.8.2, $\tau = \mu\,(du/dy)$), viscosity, $\mu$ is constant in the case of Newtonian fluids (other conditions and parameters remaining the same). The viscosity at any given temperature and pressure is constant for a Newtonian fluid and is independent of the rate of deformation. The characteristics is shown plotted in Fig. 1.8.2. Two different plots are shown as different authors use different representations.

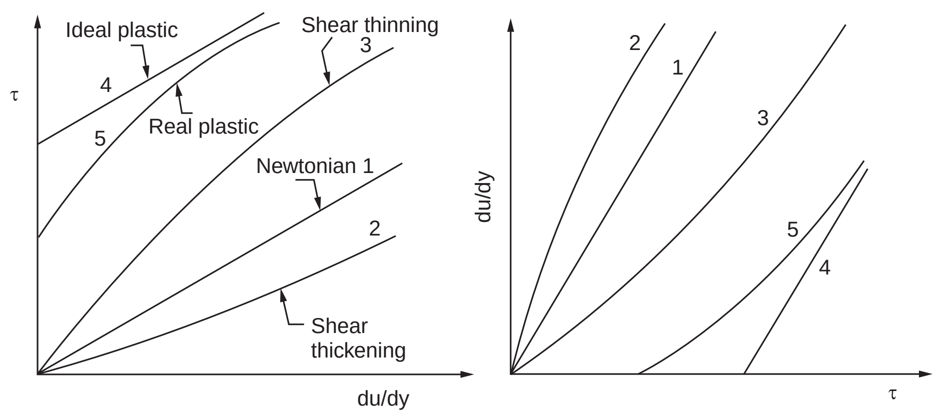



**Figure 1.8.2** Rheological behaviour of fluids

Non Newtonian fluids can be further classified as simple non Newtonian, ideal plastic and shear thinning, shear thickening and real plastic fluids. In non Newtonian fluids the viscosity will vary with variation in the rate of deformation. Linear relationship between shear stress and rate of deformation ($du/dy$) does not exist. In plastics, up to a certain value of applied shear stress there is no flow. After this limit it has a constant viscosity at any given temperature. In shear thickening materials, the viscosity will increase with ($du/dy$) deformation rate. In shear thinning materials viscosity will decrease with $du/dy$. Paint, tooth paste, printers ink are some examples for different behaviours. These are also shown in Fig. 1.8.2. Many other behaviours have been observed which are more specialised in nature. The main topic of study in this text will involve only Newtonian fluids.

### 1.8.2 Viscosity and Momentum Transfer

In the flow of liquids and gases molecules are free to move from one layer to another. When the velocity in the layers are different as in viscous flow, the molecules moving from the layer at lower speed to the layer at higher speed have to be accelerated. Similarly the molecules moving from the layer at higher velocity to a layer at a lower velocity carry with them a higher value of momentum and these are to be slowed down. Thus the molecules diffusing across layers transport a net momentum introducing a shear stress between the layers. The force will be zero if both layers move at the same speed or if the fluid is at rest.

When cohesive forces exist between atoms or molecules these forces have to be overcome, for relative motion between layers. A shear force is to be exerted to cause fluids to flow.

Viscous forces can be considered as the sum of these two, namely, the force due to momentum transfer and the force for overcoming cohesion. In the case of liquids, the viscous forces are due more to the breaking of cohesive forces than due to momentum transfer (as molecular velocities are low). In the case of gases viscous forces are more due to momentum transfer as distance between molecules is larger and velocities are higher.

### 1.8.3 Effect of Temperature on Viscosity

When temperature increases the distance between molecules increases and the cohesive force decreases. So, viscosity of liquids decrease when temperature increases.

In the case of gases, the contribution to viscosity is more due to momentum transfer. As temperature increases, more molecules cross over with higher momentum differences. Hence, in the case of gases, viscosity increases with temperature.

### 1.8.4 Significance of Kinematic Viscosity

Kinematic viscosity, $\nu = \mu/\rho$, The unit in SI system is $m^2/s$.

$$(Ns/m^2)\ (m^3/\ kg) = [(kg.m/s^2)\ (s/m^2)]\ [m^3/kg] = m^2/s$$

Popularly used unit is stoke $(cm^2/s) = 10^{-4}\ m^2/s$ named in honour of Stokes.

Centi stoke is also popular $= 10^{-6}\ m^2/s$.

Kinematic viscosity represents momentum diffusivity. It may be explained by modifying equation 1.8.2

$$\tau = \mu\ (du/dy) = (\mu/\rho) \times \{d\ (\rho u/dy)\} = \nu \times \{d\ (\rho u/dy)\} \qquad (1.8.4)$$

$d\ (\rho u/dy)$ represents momentum flux in the $y$ direction.

So, $(\mu/\rho) = \nu$ kinematic viscosity gives the rate of momentum flux or momentum diffusivity.

With increase in temperature kinematic viscosity decreases in the case of liquids and increases in the case of gases. For liquids and gases absolute (dynamic) viscosity is not influenced significantly by pressure. But kinematic viscosity of gases is influenced by pressure due to change in density. In gas flow it is better to use absolute viscosity and density, rather than tabulated values of kinematic viscosity, which is usually for 1 atm.

## 1.8.5 Measurement of Viscosity of Fluids

### 1.8.5.1 Using Flow Through Orifices

In viscosity determination using Saybolt or Redwood viscometers, the time for the flow through a standard orifice, of a fixed quantity of the liquid kept in a cup of specified dimensions is measured in seconds and the viscosity is expressed as Saybolt seconds or Redwood seconds. The time is converted to poise by empirical equations. These are the popular instruments for industrial use. The procedure is simple and a quick assessment is possible. However for design purposes viscosity should be expressed in the standard units of $Ns/m^2$.

### 1.8.5.2 Rotating Cylinder Method

The fluid is filled in the interspace between two cylinders. The outer cylinder is rotated keeping the inner cylinder stationary and the reaction torque on the inner cylinder is measured using a torsion spring. Knowing the length, diameter, film thickness, rpm and the torque, the value of viscosity can be calculated. Refer Example 1.7.

**Example 1.7.** *In a test set up as in figure to measure viscosity, the cylinder supported by a torsion spring is 20 cm in dia and 20 cm long. A sleeve surrounding the cylinder rotates at 900 rpm and the torque measured is 0.2 Nm. If the film thickness between the cylinder and sleeve is 0.15 mm,* ***determine the viscosity of the oil.***

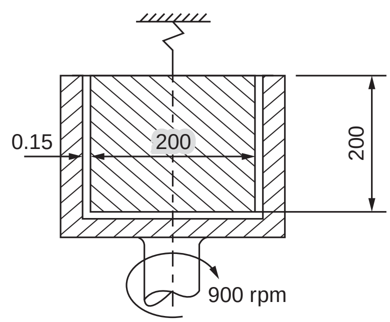



**Figure Ex. 1.7** Viscosity test setup

The total torque is given by the sum of the torque due to the shear forces on the cylindrical surface and that on the bottom surface.

Torque due to shear on the cylindrical surface (eqn 1.9.1*a*), $T_s = \mu\pi^2 NLR^3/15\ h$,

Torque on bottom surface (eqn 1.9.3),
$T_b = \mu\pi^2 NR^4/60\ h$

Where $h$ is the clearance between the sleeve and cylinder and also base and bottom. In this case both are assumed to be equal. Total torque is the sum of values given by the above equations. In case the clearances are different then $h_1$ and $h_2$ should be used.

Total torque = $(\mu\ \pi^2 NR^3/\ 15.h)\ \{L + (R/4)\}$, substituting,

$$0.2 = [(\mu \times \pi^2\ 900 \times 0.1^3)/(15 \times 0.0015)] \times [0.2 + (0.1/4)]$$

Solving for viscosity, $\boldsymbol{\mu = 0.00225\ Ns/m^2}$ **or 2.25 cP.**

This situation is similar to that in a Foot Step bearing.

### 1.8.5.3 Capillary Tube Method

The time for the flow of a given quantity under a constant head (pressure) through a tube of known diameter $d$, and length $L$ is measured or the pressure causing flow is maintained constant and the flow rate is measured.

$$\Delta P = (32\ \mu\ VL)/d^2 \quad (1.8.5)$$

This equation is known as Hagen-Poiseuille equation. The viscosity can be calculated using the flow rate and the diameter. Volume flow per second, $Q = (\pi d^2/4)\,V$. $Q$ is experimentally measured using the apparatus. The head causing flow is known. Hence μ can be calculated.

### 1.8.5.4 Falling Sphere Method

A small polished steel ball is allowed to fall freely through the liquid column. The ball will reach a uniform velocity after some distance. At this condition, gravity force will equal the viscous drag. The velocity is measured by timing a constant distance of fall.

$$\mu = 2r^2 g\ (\rho_1 - \rho_2)/9V \quad (1.8.6)$$

(μ will be in poise. 1 poise = 0.1 $Ns/m^2$)

where $r$ is the radius of the ball, $V$ is the terminal velocity (constant velocity), $\rho_1$ and $\rho_2$ are the densities of the ball and the liquid. This equation is known as **Stokes equation.**

**Example 1.8.** *Oil flows at the rate of 3 l / s through a pipe of 50 mm diameter. The pressure difference across a length of 15 m of the pipe is 6 kPa.* ***Determine the viscosity of oil flowing through the pipe.***

Using Hagen-Poiesuille equation-1.8.5 , $\Delta P = (32\ \mu u L)/d^2$

$u = Q/(\pi d^2/4) = 3 \times 10^{-3}/(\pi \times 0.05^2/4) = 1.53$ m/s

$\mu = \Delta P \times d^2/32uL = (6000 \times 0.05^2)/(32 \times 1.53 \times 15) =$ **0.0204 Ns m²**

**Example 1.9.** *A steel ball of 2 mm dia and density 8000 kg/m³ dropped into a column of oil of specific gravity 0.80 attains a terminal velocity of 2mm/s.* ***Determine the viscosity of the oil****.* *Using Stokes equation, 1.8.6*

$\mu = 2r^2 g\ (\rho_1 - \rho_2)/9u$

$= 2 \times (0.002/2)^2 \times 9.81 \times (8000 - 800)/(9 \times 0.002) = 7.85$ **Ns/m².**

## 1.9 APPLICATION OF VISCOSITY CONCEPT

### 1.9.1 Viscous Torque and Power—Rotating Shafts

Refer Figure 1.9.1

Shear stress, $\tau = \mu\ (du/dy) = \mu\ (u/y)$, as linearity is assumed

$u = \pi DN/60$, $y = h$, clearance in m

$\tau = \mu\ (\pi DN/60h)$, Tangential force $= \tau \times A$, $A = \pi DL$

Torque, $T$ = tangential force $\times D/2 = \mu\ (\pi DN/60h)\ (\pi DL)\ (D/2)$

substituting $T = \mu\ \pi^2 NLD^3/\ 120\ h$ (1.9.1)

If radius is used, $T = \mu\pi^2 NLR^3/15\ h$ (1.9.1*a*)

As power, $P = 2\pi NT/60$,

$P = \mu\pi^3 N^2 LR^3/450\ h$ (1.9.2)

For equations 1.9.1 and 1.9.2, proper units are listed below:

*L, R, D, h* should be in meter and *N* in rpm. Viscosity μ should be in $Ns/m^2$ (or Pas). The torque will be obtained in Nm and the power calculated will be in *W*.

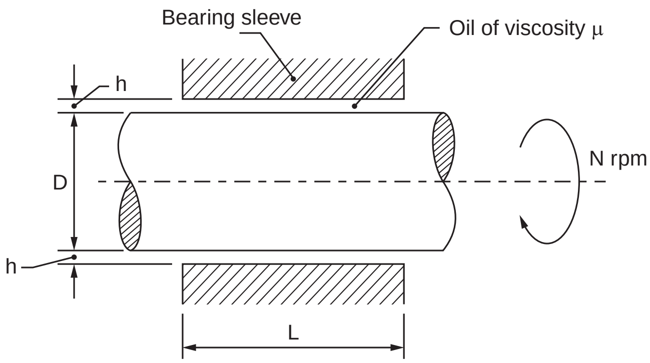



**Figure 1.9.1** Rotating Shaft in Bearing

**Note:** Clearance $h$ is also the oil film thickness in bearings. End effects are neglected. Linear velocity variation is assumed. Axial location is assumed.

**Example 1.10.** *Determine the power required to run a 300 mm dia shaft at 400 rpm in journals with uniform oil thickness of 1 mm. Two bearings of 300 mm width are used to support the shaft. The dynamic viscosity of oil is 0.03 Pas. (Pas = (N/m²) × s).*

Shear stress on the shaft surface = $\tau = \mu\ (du/dy) = \mu(u/y)$

$$u = \pi DN/60 = \pi \times 0.3 \times 400/60 = 6.28 \text{ m/s}$$

$$\tau = 0.03\ \{(6.28 - 0)/\ 0.001\} = 188.4 \text{ N/m}^2$$

Surface area of the two bearings, $A = 2\ \pi\ DL$

Force on shaft surface $= \tau \times A = 188.4 \times (2 \times \pi \times 0.3 \times 0.3) = 106.6\ N$

Torque $= 106.6 \times 0.15 = 15.995$ Nm

Power required $= 2\ \pi\ NT/60 = 2 \times \pi \times 400 \times 15.995/60 =$ **670 W.**

(check using eqn. 1.9.2, $P = \mu\ \pi^3\ N^2LR^3/450\ h = 669.74$ W)

## 1.9.2 Viscous Torque—Disk Rotating Over a Parallel Plate

Refer Figure 1.9.2.

Consider an annular strip of radius **r** and width **dr** shown in Figure 1.9.2. The force on the strip is given by,

$$F = A\mu\ (du/dy) = A\ \mu\ (u/y)$$

(as $y$ is small linear velocity variation can be assumed)

$$u = 2\ \pi rN/60,\ y = h,\ A = 2\pi r\ dr$$

Torque = Force × radius, substituting the above values

torque $dT$ on the strip is, $dT = 2\pi r\ dr\ \mu(2\pi rN/60h)r$

$$dT = 2\pi r.dr.\mu.\ 2\pi rN.r/60.h = [\mu\pi^2N/15.h]r^3dr$$

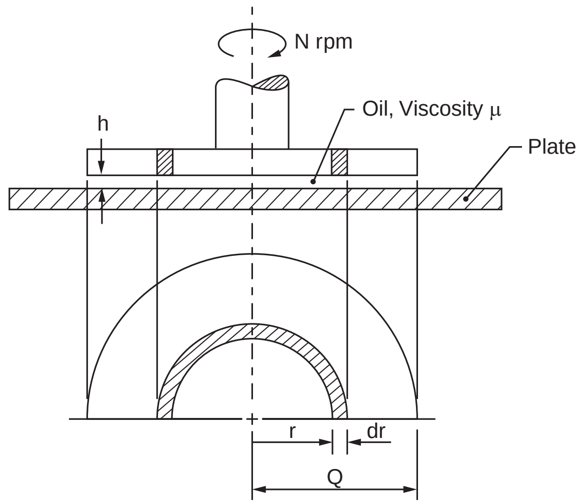



**Figure 1.9.2** Rotating disk

Integrating the expression from centre to edge *i.e.*, 0 to $R$,

$$\mathbf{T = \mu\pi^2NR^4/60\ h} \tag{1.9.3}$$

If diameter is used, $R^4 = (1/16)D^4$

$$\mathbf{T = \mu\pi^2ND^4/960\ h} \tag{1.9.3a}$$

The power required, $P = 2\pi NT/60$

$$\mathbf{P = \mu\pi^3N^2R^4/1800\ h} \tag{1.9.4}$$

use R in metre, $N$ in rpm and μ in Ns/$m^2$ or Pa s.

For an annular area like a collar the integration limits are $\mathbf{R_o}$ and $\mathbf{R_i}$ and the torque is given by

$$\mathbf{T = \mu\pi^2N(R_o{}^4 - R_i{}^4)/60\ h} \tag{1.9.5}$$

**Power,** $$\mathbf{P = \mu\pi^3N^2(R_o{}^4 - R_i{}^4)/1800\ h} \tag{1.9.6}$$

**Example 1.11. Determine the oil film thickness** *between the plates of a collar bearing of 0.2 m ID and 0.3 m OD transmitting power, if 50 W was required to overcome viscous friction while running at 700 rpm. The oil used has a viscosity of 30 cP.*

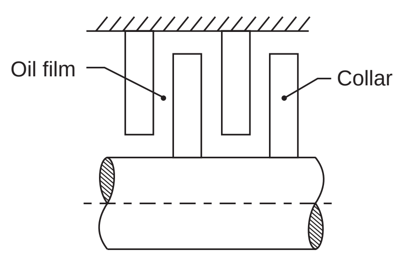



**Figure Ex. 1.11**

Power = $2\pi NT/60$ W, substituting the given values,

$50 = 2\pi \times 700 \times T/60$, Solving torque,

**T = 0.682 Nm**

This is a situation where an annular surface rotates over a flat surface. Hence, using equation 1.9.5, Torque, $T = \mu\pi^2N\ (R_O{}^4 - R_i{}^4)/60.h$

$\mu = 30$ cP = $30 \times .0001$ Ns/$m^2$, substituting the values,

$0.682 = (30 \times 0.0001) \times \pi^2 \times 700 \times (0.15^4 - 0.1^4)/60 \times h$

∴ **h = 0.000206m = 0.206 mm**

## 1.9.3 Viscous Torque—Cone in a Conical Support

Considering a small element between radius $r$ and $r + dr$, as shown in figure 1.9.3. The surface width of the element in contact with oil is

$$dx = dr/\sin\theta$$

The surface area should be calculated with respect to centre $O$ as shown in figure—the point where the normal to the surface meets the axis—or the centre of rotation, the length $OA$ being $r/\cos\theta$.

Hence contact surface area = $2\pi r.dr/\sin\theta.\cos\theta$.

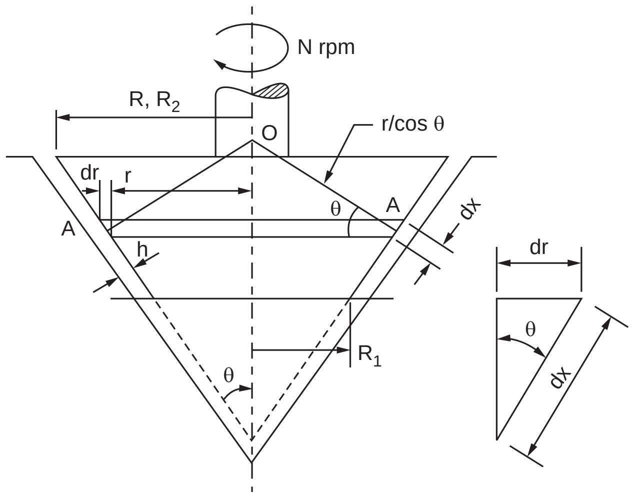



**Figure 1.9.3** Rotating cone or conical bearing

The velocity along the surface is $(2\pi rN/60).\cos\theta$ and the film thickness is $h$.

$$F = A\mu\,(du/dy) = \{(2\pi r./\sin\theta.\cos\theta)\}\,\mu(2\pi rN.\cos\theta/60)\,(1/h)$$

$$F = (\pi^2\mu Nr^2dr)/(15.h.\sin\theta), \qquad \text{Torque} = F.r$$

Torque on element, $dT = \pi^2\mu Nr^2dr.r/15.h.\sin\theta = (\pi\mu N/15\text{ h }\sin\theta)r^3\,dr$

Integrating between $r = 0$ and $r = R$

$$\mathbf{T = \pi^2\,\mu NR^4/60.h\ \sin\theta} \qquad (1.9.7)$$

Using $\mu$ in $Ns/m^2$, $h$ and $R$ in metre the torque will be in N.m. When semicone angle $\theta = 90°$, this reduces to the expression for the disk—equation 1.9.3. For contact only between $R_1$ and $R_2$.

$$\mathbf{T = \mu\pi^2\ N(R_2^{\,4} - R_1^{\,4})/60.h.\ \sin\theta} \qquad (1.9.8)$$

Power required, $$\mathbf{P = 2\pi NT/60 = \mu^3N^2[R_2^{\,4} - R_1^{\,4}]/1800\ h\ \sin\theta} \qquad (1.9.9)$$

**Exmaple 1.12.** ***Determine the power required*** *to overcome viscous friction for a shaft running at 700 rpm fitted with a conical bearing. The inner and outer radius of the conical bearing are 0.3 m and 0.5 m. The height of the cone is 0.3 m. The 1.5 mm uniform clearance between the bearing and support is filled with oil of viscosity 0.02 Ns/m*$^2$.

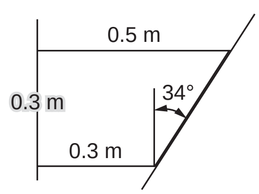



**Figure Ex. 1.12**

Equation 1.9.8 is applicable in this case.

$$\tan\theta = (0.5 - 0.3)/0.3 = 0.667, \quad \therefore \quad \theta = 34°$$

$T = \pi^2\mu\,N\,(R_o^{\,4} - R_i^{\,4})/\,60.\ h.\sin\theta$, substituting the values

$$T = \pi^2 \times 0.03 \times 700 \times (0.5^4 - 0.3^4)/60 \times 0.0015 \times \sin 34 = 149.36 \text{ Nm}$$

**Power required** $= 2\pi NT/60 = 2\pi \times 700 \times 149.36/60 =$ **10948 W**

Check using equation 1.9.9 also,

$$P = \mu \times \pi^3 \times 700^2 \times [0.5^4 - 0.3^4]/ [1800 \times 0.0015 \times \sin 34] = 10948 \text{ W}.$$

Note the high value of viscosity

## 1.10 SURFACE TENSION

Many of us would have seen the demonstration of a needle being supported on water surface without it being wetted. This is due to the surface tension of water.

All liquids exhibit a free surface known as meniscus when in contact with vapour or gas. Liquid molecules exhibit cohesive forces binding them with each other. The molecules below the surface are generally free to move within the liquid and they move at random. When they reach the surface they reach a dead end in the sense that no molecules are present in great numbers above the surface to attract or pull them out of the surface. So they stop and return back into the liquid. A thin layer of few atomic thickness at the surface formed by the cohesive bond between atoms slows down and sends back the molecules reaching the surface. This cohesive bond exhibits a tensile strength for the surface layer and this is known as surface tension. Force is found necessary to stretch the surface.

Surface tension may also be defined as the work in Nm/m$^2$ or N/m required to create unit surface of the liquid. The work is actually required for pulling up the molecules with lower energy from below, to form the surface.

Another definition for surface tension is the force required to keep unit length of the surface film in equilibrium (N/m). The formation of bubbles, droplets and free jets are due to the surface tension of the liquid.

### 1.10.1 Surface Tension Effect on Solid-Liquid Interface

In liquids cohesive forces between molecules lead to surface tension. The formation of droplets is a direct effect of this phenomenon. So also the formation of a free jet, when liquid flows out of an orifice or opening like a tap. The pressure inside the droplets or jet is higher due to the surface tension.

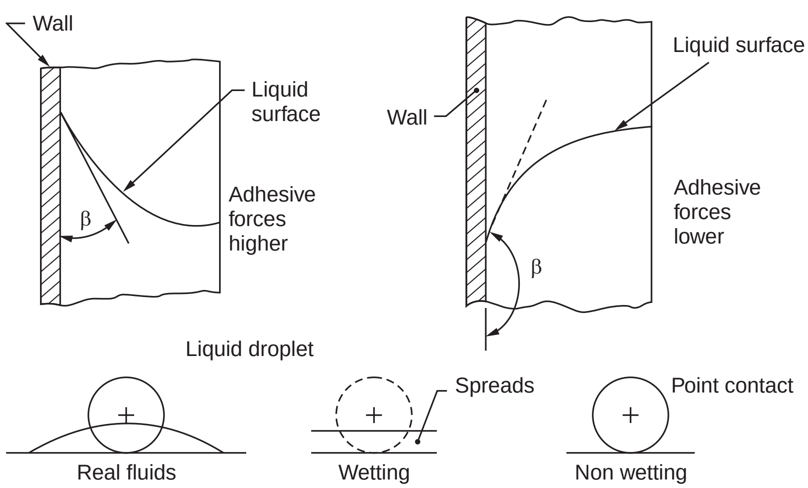



**Figure 1.10.1** Surface tension effect at solid-liquid interface

Liquids also exhibit adhesive forces when they come in contact with other solid or liquid surfaces. At the interface this leads to the liquid surface being moved up or down forming a curved surface. When the adhesive forces are higher the contact surface is lifted up forming a concave surface. Oils, water etc. exhibit such behaviour. These are said to be surface wetting. When the adhesive forces are lower, the contact surface is lowered at the interface and a convex surface results as in the case of mercury. Such liquids are called nonwetting. These are shown in Fig. 1.10.1.

The angle of contact "β" defines the concavity or convexity of the liquid surface. It can be shown that if the surface tension at the solid liquid interface (due to adhesive forces) is $\sigma_{s1}$ and if the surface tension in the liquid (due to cohesive forces) is $\sigma_{11}$ then

$$\cos \beta = [(2\sigma_{s1}/\sigma_{11}) - 1] \quad (1.10.1)$$

At the surface this contact angle will be maintained due to molecular equilibrium. The result of this phenomenon is capillary action at the solid liquid interface. The curved surface creates a pressure differential across the free surface and causes the liquid level to be raised or lowered until static equilibrium is reached.

**Example 1.13.** ***Determine the surface tension*** *acting on the surface of a vertical thin plate of 1m length when it is lifted vertically from a liquid using a force of 0.3N.*

Two contact lines form at the surface and hence, Force = 2 × 1 × Surface tension

0.3 = 2 × 1 × Surface tension. Solving, Surface tension, **σ = 0.15 N/m.**

### 1.10.2 Capillary Rise or Depression

Refer Figure 1.10.2.

Let $D$ be the diameter of the tube and β is the contact angle. The surface tension forces acting around the circumference of the tube = $\pi \times D \times \sigma$.

The vertical component of this force = $\pi \times D \times \sigma \times \cos\beta$

This is balanced by the fluid column of height, $h$, the specific weight of liquid being γ.

Equating, $h \times \gamma \times A = \pi \times D \times \sigma \cos\beta$, $A = \pi D^2/4$ and so

$$h = (4\pi \times D \times \sigma \times \cos\beta)/(\gamma\pi D^2) = (4\sigma \times \cos\beta)/\rho g D \quad (1.10.2)$$

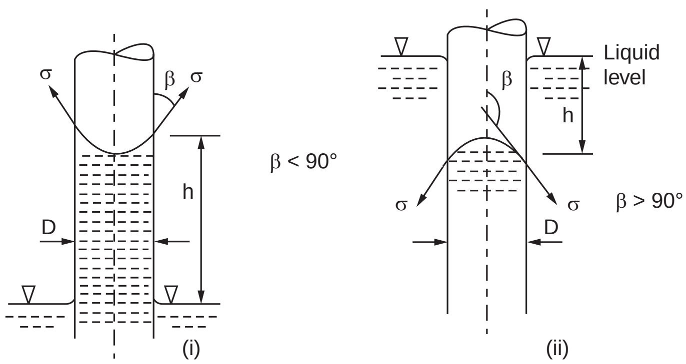



**Figure 1.10.2** Surface tension, (*i*) capillary rise (*ii*) depression

This equation provides the means for calculating the capillary rise or depression. The sign of cos β depending on β > 90 or otherwise determines the capillary rise or depression.

**Example 1.14.** ***Determine the capillary depression*** *of mercury in a 2 mm ID glass tube. Assume $\sigma = 0.5$ N/m and $\beta = 130°$.*

Specific weight of mercury, $\gamma = 13600 \times 9.81$ N/m$^3$

Using eqn. 1.10.2, $h = (4\,\sigma \times \cos\beta)/\rho g/D$

$= (4 \times 0.5 \times \cos 130)/(13600 \times 9.81 \times 0.002)$

$= -4.82 \times 10^{-3}$ m = **– 4.82 mm**

**Example1.15.** *In a closed end single tube manometer, the height of mercury column above the mercury well shows 757 mm against the atmospheric pressure. The ID of the tube is 2 mm. The contact angle is 135°.* ***Determine the actual height*** *representing the atmospheric pressure if surface tension is 0.48 N/m. The space above the column may be considered as vacuum.*

Actual height of mercury column = Mercury column height + Capillary depression

Specific weight of mercury $= \rho g = 13600 \times 9.81$ N/m$^3$

Capillary depression, $\mathbf{h} = (4\,\sigma \times \cos\beta)/\gamma D$

$= (4 \times 0.48 \times \cos 135)/(0.002 \times 13600 \times 9.81)$

$= -5.09 \times 10^{-3}$m **= – 5.09 mm (depression)**

Corrected height of mercury column = 757 + 5.09 = **762.09 mm**

### 1.10.3 Pressure Difference Caused by Surface Tension on a Doubly Curved Surface

Consider the small doubly curved element with radius $r_1$ and included angle $d\phi$ in one direction and radius $r_2$ and $d\theta$ in the perpendicular direction referred to the normal at its center.

For equilibrium the components of the surface tension forces along the normal should be equal to the pressure difference.

The sides are $r_1\, d\phi$ and $r_2\, d\theta$ long. Components are $\sigma r_1 \sin(d\theta/2)$ from θ direction sides and $\sigma r_2 \sin(d\phi/2)$ from the φ direction sides.

$$2\sigma r_1 d\phi \sin(d\theta/2) + 2\sigma r_2\, d\theta \sin(d\phi/2) = (p_i - p_o) r_1 r_2\, d\theta d\phi$$

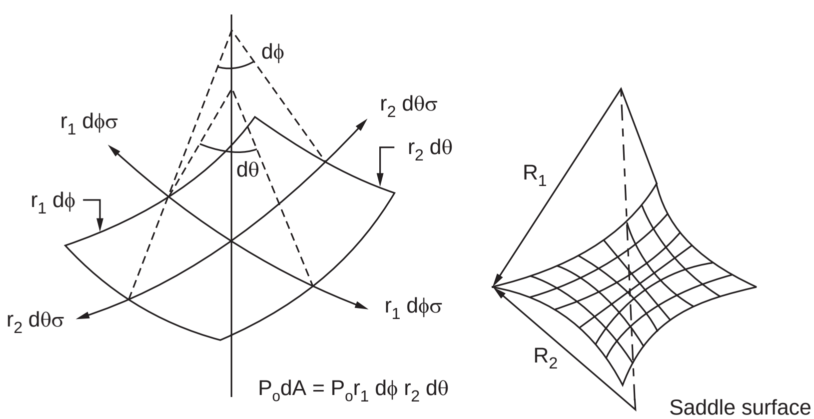



**Figure 1.10.3** Pressure difference, doubly curved surface

For small values of angles, sin θ = θ, in radians. Cancelling the common terms

$$\sigma [r_1 + r_2] = (p_i - p_o) \times r_1 r_2. \text{ Rearranging,} \quad (1.10.3)$$

$$\mathbf{(p_i - p_o) = [(1/r_1) + (1/r_2)] \times \sigma}$$

For a spherical surface, $r_1 = r_2 = R$

So, $$\mathbf{(p_i - p_o) = 2\sigma/R} \quad (1.10.4)$$

where $R$ is the radius of the sphere.

For cylindrical shapes one radius is infinite, and so

$$\mathbf{(p_i - p_o) = \sigma/R} \quad (1.10.4a)$$

These equations give the pressure difference between inside and outside of droplets and free jets of liquids due to surface tension. The pressure inside air bubbles will be higher compared to the outside pressure. The pressure inside a free jet will be higher compared to the outside.

The pressure difference can be made zero for a doubly curved surface if the curvature is like that of a saddle (one positive and the other negative). This situation can be seen in the jet formed in tap flow where internal pressure cannot be maintained.

## 1.10.4 Pressure Inside a Droplet and a Free Jet

Refer Figure 1.10.4.

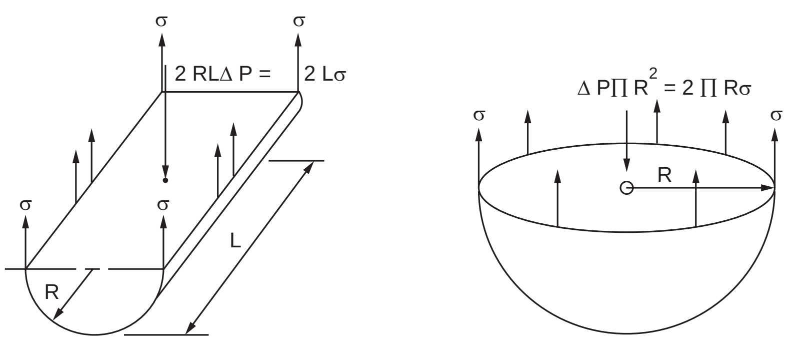



**Figure 1.10.4** Surface tension effects on bubbles and free jets

Considering the sphere as two halves or hemispheres of diameter $D$ and considering the equilibrium of these halves,

Pressure forces = Surface tension forces, $(p_i - p_o)(\pi D^2/4) = \sigma \times \pi \times D$

$$\mathbf{(p_i - p_o) = 4(\sigma/D) = 2(\sigma/R)} \quad (1.10.5)$$

Considering a cylinder of length $L$ and diameter $D$ and considering its equilibrium, taking two halves of the cylinder.

pressure force = $DL(p_i - p_o)$, surface tension force = $2\sigma L$

$$\mathbf{(p_i - p_o) = 2 (\sigma/D) = (\sigma/R)} \quad (1.10.6)$$

**Example 1.16.** ***Determine the pressure difference*** *across a nozzle if diesel is sprayed through it with an average diameter of 0.03mm.* The surface tension is 0.04N/m.

The spray is of cylindrical shape

$$\mathbf{P} = \sigma/R = 0.04/(0.03 \times 10^{-3}/2) = 2666.67 \text{ N/m}^2 = \mathbf{2.67\ kpa}$$

**Example 1.17.** ***Calculate the surface tension*** *if the pressure difference between the inside and outside of a soap bubble of 3mm dia is 18 N/m².*

Referring equation 1.10.5, $\Delta P = 4\sigma/D$

Surface tension, $\sigma = \Delta P \times D/4 = 18 \times (0.003/4) = \mathbf{0.0135\ N/m}$

## 1.11 COMPRESSIBILITY AND BULK MODULUS

Bulk modulus, $E_v$ is defined as the ratio of the change in pressure to the rate of change of volume due to the change in pressure. It can also be expressed in terms of change of density.

$$\mathbf{E_v = -\,dp/(dv/v) = dp/(d\rho/\rho)} \quad (1.11.1)$$

where **dp** is the change in pressure causing a change in volume **dv** when the original volume was **v**. The unit is the same as that of pressure, obviously. Note that $dv/v = -\,d\rho/\rho$.

The negative sign indicates that if $dp$ is positive then $dv$ is negative and vice versa, so that the bulk modulus is always positive (N/m²). The symbol used in this text for bulk modulus is $E_v$ ($K$ is more popularly used).

This definition can be applied to liquids as such, without any modifications. In the case of gases, the value of compressibility will depend on the process law for the change of volume and will be different for different processes.

The bulk modulus for liquids depends on both pressure and temperature. The value increases with pressure as $dv$ will be lower at higher pressures for the same value of $dp$. With temperature the bulk modulus of liquids generally increases, reaches a maximum and then decreases. For water the maximum is at about 50°C. The value is in the range of 2000 MN/m² or $2000 \times 10^6$ N/m² or about 20,000 atm. Bulk modulus influences the velocity of sound in the medium, which equals $(g_o \times E_v/\rho)^{0.5}$.

**Example 1.18.** ***Determine the bulk modulus of*** *a liquid whose volume decreases by 4% for an increase in pressure of 500 × 10⁵ pa. Also* ***determine the velocity of sound*** *in the medium if the density is 1000 kg/m³.*

Bulk modulus is defined as $E_v = -\,dp/(dv/v)$, substituting the values,

$$E_v = (-\,500 \times 10^5)/(-4/100) = \mathbf{1.25 \times 10^9\ N/m^2}$$

Velocity of sound $c$ is defined as $= (g_o \times E_v/\rho)^{0.5}$

$$\therefore \quad \mathbf{c} = [1 \times 1.25 \times 10^9/100]^{0.05} = \mathbf{1118\ m/s.}$$

**Example 1.19.** *The pressure of water in a power press cylinder is released from 990 bar to 1 bar isothermally. If the average value of bulk modulus for water in this range is 2430 × 10⁶ N/m².* ***What will be the percentage increase in specific volume?***

The definition of bulk modulus, $E_v = -\,dp/(dv/v)$ is used to obtain the solution. Macroscopically the above equation can be modified as

$$E_v = -\,\{P_1 - P_2\}\{(v_2 - v_1)/v_1\}, \text{ Rearranging,}$$

Change in specific volume $= (v_2 - v_1)/v_1 = -(P_2 - P_1)/E_v$

$= (990 \times 10^5 - 1 \times 10^5)/2430 \times 10^6 = 0.0407$

**% change in specific volume = 4.07%**

**Example 1.20.** *Density of sea water at the surface was measured as 1040 kg/m³ at an atmospheric pressure of 1 bar. At certain depth in water, the density was found to be 1055 kg/m³.* ***Determine the pressure at that point.*** *The bulk modulus is 2290 × 10⁶ N/m².*

Bulk modulus, $E_v = -dp/(dv/v) = -(P_2 - P_1)/[(v_2 - v_1)v_1]$

As $v = 1/\rho$, $-(P_2 - P_1) = E_v \times [\{1/\rho_2) - (1/\rho_1)\}/(1/\rho_1)]$

$= E_v \times [(\rho_1 - \rho_2)/\rho_2]$

$P_2 = P_1 - E_v \times [(\rho_1 - \rho_2)/\rho_2] = 1 \times 10^5 - 2290 \times 10^6 \{(1040 - 1055)/1055\}$

$=$ **$32.659 \times 10^6$ N/m² or about 326.59 bar.**

### 1.11.1 Expressions for the Compressibility of Gases

The expression for compressibility of gases for different processes can be obtained using the definition, namely, **compressibility = – dp/(dv/v)**. In the case of gases the variation of volume, **dv**, with variation in pressure, **dp**, will depend on the process used. The relationship between these can be obtained using the characteristic gas equation and the equation describing the process.

Process equation for gases can be written in the following general form

$$\mathbf{Pv^n = constant} \qquad (1.11.2)$$

**where n can take values from 0 to** ∞. If $n = 0$, then $P$ = constant or the process is a constant pressure process. If $n = \infty$, then $v$ = constant and the process is constant volume process. These are not of immediate interest in calculating compressibility. If $dp = 0$, compressibility is zero and if $dv = 0$, compressibility is infinite.

The processes of practical interest are for values of $n = 1$ to $n = c_p/c_v$ (the ratio of specific heats, denoted as $k$). The value $n = 1$ means $Pv$ = constant or isothermal process and $n = c_p/c_v = k$ means isentropic process.

Using the equation $Pv^n$ = constant and differentiating the same,

$$\mathbf{nPv^{(n-1)}dv + v^n dp = 0} \qquad (1.11.3)$$

rearranging and using the definition of $E_v$,

$$\mathbf{E_v = -dp/(dv/v) = n \times P} \qquad (1.11.4)$$

Hence compressibility of gas varies as the product $n \times P$.

For isothermal process, $n = 1$, compressibility $= P$.

For isentropic process, compressibility $= k \times P$.

For constant pressure and constant volume processes compressibility values are zero and ∞ respectively.

In the case of gases the velocity of propagation of sound is assumed to be isentropic. From the definition of velocity of sound as $[g_o \times E_v/\rho]^{0.5}$ it can be shown that

$$\mathbf{c = [g_o \times k\, P/\rho]^{0.5} = [g_o \times k \times R \times T]^{0.5}} \qquad (1.11.5)$$

It may be noted that for a given gas the velocity of sound depends only on the temperature. As an exercise the velocity of sound at 27°C for air, oxygen, nitrogen and hydrogen may be calculated as 347.6 m/s, 330.3 m/s, 353.1 m/s and 1321.3 m/s.

## 1.12 VAPOUR PRESSURE

Liquids exhibit a free surface in the container whereas vapours and gases fill the full volume. Liquid molecules have higher cohesive forces and are bound to each other. In the gaseous state the binding forces are minimal.

Molecules constantly escape out of a liquid surface and an equal number constantly enter the surface when there is no energy addition. The number of molecules escaping from the surface or re-entering will depend upon the temperature.

Under equilibrium conditions these molecules above the free surface exert a certain pressure. This pressure is known as vapour pressure corresponding to the temperature. As the temperature increases, more molecules will leave and re-enter the surface and so the vapour pressure increases with temperature. All liquids exhibit this phenomenon. Sublimating solids also exhibit this phenomenon.

The vapour pressure is also known as saturation pressure corresponding to the temperature. The temperature corresponding to the pressure is known as saturation temperature. If liquid is in contact with vapour both will be at the same temperature and under this condition these phases will be in equilibrium unless energy transaction takes place.

The vapour pressure data for water and refrigerants are available in tabular form. The vapour pressure increases with the temperature. For all liquids there exists a pressure above which there is no observable difference between the two phases. This pressure is known as critical pressure. Liquid will begin to boil if the pressure falls to the level of vapour pressure corresponding to that temperature. Such boiling leads to the phenomenon known as cavitation in pumps and turbines. In pumps it is usually at the suction side and in turbines it is usually at the exit end.

### 1.12.1 Partial Pressure

In a mixture of gases the total pressure $P$ will equal the sum of pressures exerted by each of the components if that component alone occupies the full volume at that temperature. The pressure exerted by each component is known as its partial pressure.

$$\mathbf{P = p_1 + p_2 + p_3 + \ldots.} \tag{1.12.1}$$

where $p_1 = (m_1R_1T)/V$ ; $p_2 = (m_2R_2T)/V$ in which $T$ and $V$ are the common temperature and volume.

For example air is a mixture of various gases as well as some water vapour. The atmospheric pressure is nothing but the sum of the pressures exerted by each of these components. Of special interest in this case is the partial pressure of water vapour. This topic is studied under Psychrometry. The various properties like specific heat, gas constant etc. of the mixture can be determined from the composition.

$$\mathbf{c_m = \Sigma (c_i \times m_i)/\Sigma m_i} \tag{1.12.2}$$

where $c_m$ is the specific heat of the mixture and $c_i$ and $m_i$ are the specific heat and the mass of component $i$ in the mixture.

## SOLVED PROBLEMS

**Problem 1.1.** *A liquid with kinematic viscosity of 3 centi stokes and specific weight 9 kN/m$^3$ fills the space between a large stationary plate and a parallel plate of 475 mm square, the film thickness being 1 mm. If the smaller plate is to be pulled with uniform velocity of 4 m/s,* ***determine the force required*** *if the liquid film is maintained all through.*

The force required (eqn 1.8.2), $\tau \times A = A \times \mu \times (du/dy)$, where $\tau$ is shear stress, and $\mu$ is dynamic viscosity. In this problem kinematic viscosity and specific weight are given.

Stoke = $10^{-4}$ m$^2$/s. Density = specific weight/$g$. So, $\mu = 0.03 \times 10^{-4} \times 9000/9.81$ Ns/m$^2$

**Force** = $[0.03 \times 10^{-4} \times 9000/9.81] \times (4.0/0.001) \times 0.475 \times 0.475$ = **2.484 N.**

**Problem 1.2.** *A small thin plane surface is pulled through the liquid filled space between two large horizontal planes in the parallel direction. Show that the force required will be minimum if the plate is located midway between the planes.*

Let the velocity of the small plane be $u$, and the distance between the large planes be $h$.

Let the small plane be located at a distance of $y$ from the bottom plane. Assume linear variation of velocity and unit area. Refer Fig. P 1.2.

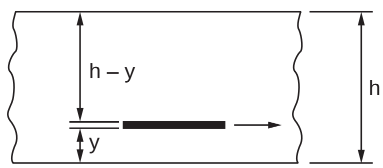


**Figure P.1.2** Problem model

Velocity gradient on the bottom surface = $u/y$

Velocity gradient on the top surface = $u/(h - y)$,

Considering unit area,

Force on the bottom surface = $\mu \times (u/y)$, Force on the top surface = $\mu \times u/(h - y)$

Total force to pull the plane = $\mu \times u \times \{(1/y) + [1/(h - y)]\}$ ...(A)

To obtain the condition for minimisation of the force the variation of force with respect to $y$ should be zero, or $dF/dy = 0$, Differentiating the expression $A$,

$$dF/dy = \mu \times u \ \{(-1/y^2) + [1/(h - y)^2]\}, \text{ Equating to zero}$$

$$y^2 = (h - y)^2 \text{ or } y = h/2$$

or the plane should be located at the mid gap position for the force to be minimum.

The force required for different location of the plate is calculated using the following data and tabulated below.

$$\mu = 0.014 \text{ Ns/m}^2, u = 5 \text{ m/s}, h = 0.1 \text{ m.}$$

Equation A is used in the calculation.

Model calculation is given for $y$ = 0.002 m.

$$F = 0.014 \times 5 \times \{(1/0.002) + [1/0.01 - 0.002)]\} = 43.75 \text{ N/m}^2$$

Note that the minimum occurs at mid position

| Distance, $y$ mm | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| Force, N/m$^2$ | 43.75 | 33.33 | 29.17 | 28.00 | 29.17 |

**Problem 1.3.** *A small plane is pulled along the centre plane of the oil filled space between two large horizontal planes with a velocity u and the force was measured as F. The viscosity of the oil was* $\mu_1$. *If a lighter oil of viscosity* $\mu_2$ *fills the gap what should be the location of the plate for the force to be the same when pulled with the same velocity u.*

If the plane is located centrally in the case where the oil is lighter the force will be smaller.

So the plane should now be located away from the central plane. Let it be located at a distance, $y$ from the lower plane as shown in Fig. P1.2 :

**Case 1:** The velocity gradient is equal on both sides = $u/(h/2) = 2 \times u/h$

Total force = $\mu_1 \times \{(2u/h) + (2u/h)\} = \mathbf{4 \times \mu_1 \times u/h}$

**Case 2:** Velocity gradient on the top surface = $u/(h-y)$

Velocity gradient on the bottom surface = $u/y$

Total force = $\mu_2 \times u \times \{(1/y) + [1/(h-y)]\} = \mu_2 \times u \times \{h/[y \times (h-y)]\}$

Equating and solving, $(\mu_2/\mu_1) = 4 \times y \times (h-y)/h^2 = \mathbf{4[y/h] \times [1 - (y/h)]}$

Solve for $(y/h)$. A quadratic equation.

**Problem 1.4.** *A large thin plate is pulled through a narrow gap filled with a fluid of viscosity μ on the upper side and a fluid of viscosity cμ on the lower side. Derive an expression for the location of the plate in the gap for the total force to be minimum.*

The force will not be minimum if the plate is centrally located as the viscosity are not equal. Let the plate be located at a distance of $y$ from the lower surface on the side where the viscosity is $c\mu$. Let the gap size be $h$, the total force for unit area will be

$$F = c\mu \times (u/y) + \mu \times u/(h-y) = \mu \times u\ \{(c/y) + [1/(h-y)]\}$$

At the minimum conditions the slope *i.e.*, the derivative $dF/dy$ should be zero.

$$dF/dy = \mu \times u\ \{[1/(h-y)^2] - [c/y^2]\}, \text{ Equating to zero yields, } y^2 = c \times (h-y)^2$$

Taking the root,

$$\sqrt{c} \times (h-y) = y \quad \text{or} \quad y = (h \times \sqrt{c})/(1 + \sqrt{c}) = h/[1 + (1/\sqrt{c})]$$

Consider the following values for the variables and calculate the force for different locations of the plate.

$$u = 5 \text{ m/s},\ \mu = 0.014 \text{ N/m}^2,\ h = 4 \text{ mm} \quad \text{and} \quad c = 0.49 \quad \text{or} \quad \sqrt{c} = 0.7$$

For optimum conditions

$$y = (0.004 \times 0.7)/(1 + 0.7) = 0.001647 \text{ m}$$

Using $F = 5 \times 0.014 \times \{(0.49/y) + [1/(0.004 - y)]\}$, the force for various locations is calculated and tabulated below:

| $y$, mm | 1.0 | 1.5 | 1.65 | 2.0 | 2.5 |
|---|---|---|---|---|---|
| Force, N/m$^2$ | 57.63 | 50.87 | 50.58 | 52.15 | 60.39 |

**Problem 1.5.** *A hydraulic lift shaft of 450 mm dia moves in a cylinder of 451 mm dia with the length of engagement of 3 m. The interface is filled with oil of kinematic viscosity of $2.4 \times 10^{-4}$ m²/s and density of 900 kg/m³. Determine the uniform velocity of movement of the shaft if the drag resistance was 300 N.*

The force can be determined assuming that the sliding is between the developed surfaces, the area being $\pi \times D \times L$, $\mu = \rho v = 2.4 \times 10^{-4} \times 900 = 0.216$ Ns/m²,

Clearance = $(D_o - D_i)/2 = 0.5$ mm. Using equations 1.8.1 and 1.8.2

Drag resistance = 300 = $\mu \times 0.45 \times 3 \times 0.216 \times (u / 0.0005)$

Solving for $u$, velocity = **0.16374 m/s.**

**Problem 1.6.** *A shaft of 145 mm dia runs in journals with a uniform oil film thickness of 0.5 mm. Two bearings of 20 cm width are used. The viscosity of the oil is 19 cP.* ***Determine the speed if the power absorbed is 15 W.***

The equation that can be used is, 1.9.2 *i.e.,* ($n$ is used to denote rpm)

$$P = [\mu\pi^3 n^2 L R^3 / 450\, h]$$

The solution can be obtained from basics also. Adopting the second method,

$$\tau = \mu\,(du/dy) = \mu\,(u/y),\ \mu = 19 \text{ cP} = 0.019 \text{ Ns/m}^2,$$

$$y = 0.5 \text{ mm} = 0.0005 \text{ m, let the rpm be } n$$

$$u = \pi Dn / 60 = \pi \times 0.145 \times n/60 = 7.592 \times 10^{-3} \times n$$

$$\tau = 0.019\,(7.592 \times 10^{-3} \times n/0.0005) = 0.2885 \times n \text{ N/m}^2,$$

$$A = 2 \times \pi DL = 0.182 \text{ m}^2, \text{ Force F} = \text{A} \times \tau = 0.2885 \times n \times 0.182 = 0.0525 \times n,$$

Torque = force × radius,

$$T = 0.0525 \times n \times 0.145/2 = 3.806 \times 10^{-3} \times n \text{ Nm}$$

Power, $P = 2\pi nT/60 = 15 = 2 \times \pi \times n\ 3.806 \times 10^{-3} \times n / 60$

Solving, speed, $n$ = **194 rpm.** (Check using the equation 1.9.2)

$$15 = [0.019 \times \pi^3 \times n^2\,(2 \times 0.20) \times 0.0725^3 / (450 \times 0.0005)]$$

Solving speed, **n = 194 rpm.**

**Problem 1.7.** *A circular disc of 0.3 m dia rotates over a large stationary plate with 1 mm thick fluid film between them.* ***Determine the viscosity of the fluid*** *if the torque required to rotate the disc at 300 rpm was 0.1 Nm.*

The equation to be used is 1.9.3, ($n$ denoting rpm)

Torque $T = (\mu \times \pi^2 \times n \times R^4)/(60 \times h)$, ($h$ – clearance),

$n = 300$ rpm, $R = 0.15$ m, $h = 0.001$ m, Substituting the values,

$$0.1 = \mu \times \pi^2 \times 300 \times 0.15^4 / (60 \times 0.001), \text{ Solving for } \mu$$

Viscosity $\boldsymbol{\mu = 4 \times 10^{-3}}$ **Ns/m² or 4 cP.**

(care should be taken to use radius value, check from basics.)

**Problem 1.8.** ***Determine the viscous drag torque and power*** *absorbed on one surface of a collar bearing of 0.2 m ID and 0.3 m OD with an oil film thickness of 1 mm and a viscosity of 30 cP if it rotates at 500 rpm.*

The equation applicable is 1.9.5. $T = \mu \times \pi^2 \times n \times (R_o^{\,4} - R_i^{\,4})/60 \times h$

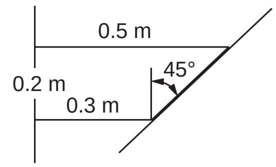
$$\mu = 30 \times 0.001 \text{ Ns/m}^2,\ n = 500 \text{ rpm},\ R_o = 0.15 \text{ m},\ R_i = 0.1 \text{ m},\ h = 0.002 \text{ m}$$

substituting the values

$$\mathbf{T} = 30 \times 0.001 \times \pi^2 \times 500 \times \{0.15^4 - 0.1^4\}/\{(60 \times 0.002)\} = \mathbf{0.5012\ Nm}$$

$$\mathbf{P} = 2\pi nT/60 = 2 \times \pi \times 500 \times 0.5012/60 = \mathbf{26.243\ W.}$$

**Problem 1.9.** *A conical bearing of outer radius 0.5 m and inner radius 0.3 m and height 0.2 m runs on a conical support with a uniform clearance between surfaces. Oil with viscosity of 30 cP is used. The support is rotated at 500 rpm.* ***Determine the clearance*** *if the power required was 1500 W.*




The angle θ is determined using the difference in radius and the length.

$$\tan \theta = (0.5 - 0.3)/0.2 = 1.0;\ \text{So } \theta = 45°.$$

Using equation 1.9.9 *i.e.*,

$$P = \pi^3 \times \mu \times n^2 \times (R_2^{\,4} - R_1^{\,4})/1800 \times h \times \sin\theta$$

$$(\mu = 30 \text{ cP} = 0.03 \text{ Ns/m}^2,\ n = 500 \text{ rpm},\ R_2 = 0.5 \text{ m},\ R_1 = 0.3 \text{ m})$$

$$1500 = \pi^3 \times 0.03 \times 500^2 \times (0.5^4 - 0.3^4)/1800 \times h \times \sin 45°$$

Solving for clearance, $\mathbf{h = 6.626 \times 10^{-3}}$ **m or 6.63 mm**

**Problem 1.10.** If the variation of velocity with distance from the surface, $y$ is given by $u = 10\, y^{0.5}$ *whre u is in m/s and y is in m in a flow field up to y = 0.08 m,* **determine the wall shear stress** *and the shear stress at y = 0.04 and 0.08 m from the surface.*

$$u = 10y^{0.5},\ (du/dy) = 5/y^{0.5}.$$

The substitution $y = 0$ in the above will give division by zero error. It has to be approximated as $(u_2 - u_1)/(y_2 - y_1)$ for near zero values of $y$.

Considering layers $y = 0$ and $y = 10^{-6}$, the velocities are 0.0 and 0.01 m/s

(using $u = 10\, y^{0.5}$), the difference in $y$ value is $10^{-6}$.

So $(u_2 - u_1)/(y_2 - y_1) = 0.01/10^{-6} = 10000$,

**At the wall,** $(du/dy) = 10000$, $\tau = \mu\,(du/dy) = \mathbf{10000 \times \mu}$

**At** $\mathbf{y = 0.04,}$ $(du/dy) = 5/0.04^{0.5} = 25$, $\tau = \mathbf{25 \times \mu}$

**At** $\mathbf{y = 0.08,}$ $(du/dy) = 5/0.08^{0.5} = 17.68$, $\tau = \mathbf{17.68 \times \mu}$

In this case the clearance considered is large and so the assumption of linear velocity variation may lead to larger error. The concept that the torque along the radius should be constant can be used to determine the torque more accurately.

**Problem 1.11.** *A hollow cylinder of 12 cm ID filled with fluid of viscosity 14 cP rotates at 600 rpm. A shaft of diameter 4 cm is placed centrally inside.* ***Determine the shear stress*** *on the shaft wall.*

The hollow cylinder rotates while the shaft is stationary. Shear stress is first calculated at the hollow cylinder wall (Assume 1 m length).

Solution is obtained from basics. Linear velocity variation is assumed. Clearance,

$$h = 0.04 \text{ m},\ \mu = 14 \times 0.001 = 0.014 \text{ N/m}^2$$

At the inside wall of the hollow cylinder,

$$u = 2\,\pi Rn/60 = 3.77 \text{ m/s}$$

$$(du/dr) = u/h = 3.77/0.04 = 94.25/\text{s},\ \tau = \mu\,(du/dr)$$

$$= 0.014 \times 94.25 = 1.32 \text{ N/m}^2$$

$$F = \pi \times D \times L \times \tau = \pi \times 0.12 \times 1 \times 1.31 = 0.498 \text{ N}$$

$$\text{torque} = F \times R = 0.498 \times 0.06 = 29.86 \times 10^{-3} \text{ Nm}$$

Torque at all radii should be the same. At mid radius $R = 0.04$ m, the velocity gradient is obtained by using this concept.

$$29.86\ 10^{-3} = \left.\frac{du}{dr}\right|_{0.04} \times 0.014 \times \pi \times 0.08 \times 1 \times 0.04,$$

Solving, $$\left.\frac{du}{dr}\right|_{0.04} = 212.06/\text{s},$$

This can be checked using equation, (see problem 1.13)

$$\left.\frac{du}{dr}\right|_{R_1} = \left.\frac{du}{dr}\right|_{R_2} \times (R_2^{\,2}/R_1^{\,2}) \text{ at } 0.04,\ \left.\frac{du}{dr}\right|_{0.04} \times 25 \times 0.06^2/0.04^2 = 212.06/\text{s}$$

The velocity gradient at the shaft surface = $94.25 \times 0.06^2/0.02^2 = 848.25$/s

**Shear stress at the shaft wall** = 848.25 × 0.014 = **11.88 N/m².**

**Problem 1.12.** *The velocity along the radius of a pipe of 0.1 m radius varies as $u = 10 \times [1 - (r/0.1)^2]$ m/s. The viscosity of the fluid is 0.02 Ns/m$^2$.* ***Determine the shear stress*** *and the shear force over the surface at r = 0.05 and r = 0.1 m.*

$$\tau = \mu\,(du/dr),\ u = 10 \times [1 - (r/0.1)^2],$$

$$du/dr = -\,10 \times (2 \times r/0.1^2) = -\,2000\,r$$

(the –ve sign indicates that the force acts opposite to the flow direction.)

$$\tau = 0.02 \times (-\,2000) \times r = -\,40\,r,\ \text{Shear force } F = 2\pi rL\tau,\ \text{Considering } L = 1$$

**At r = 0.05, τ = – 2 N/m², F = 0.628 N**

**At r = 0.1, τ = – 4 N/m², F = 2.513 N.**

**Problem 1.13.** *A sleeve surrounds a shaft with the space between them filled with a fluid. Assuming that when the sleeve rotates velocity gradient exists only at the sleeve surface and when the shaft rotates velocity gradient exists only at the shaft surface, determine the ratio of these velocity gradients.*

The torque required for the rotation will be the same in both cases. Using general notations,

$$\tau_i\,[2\pi\,r_i \times L] \times r_i = \tau_o\,[2\pi\,r_o \times L] \times r_o$$

$$\tau_i = \mu\,(du/dr)_{ri},\ \tau_o = \mu\,(du/dr)_{ro}$$

Substituting in the previous expression and solving

$$(du/dr)_i = (du/dr)_o \times [r_o^{\,2}/r_i^{\,2}]$$

This will plot as a second degree curve. When the gap is large % error will be high if linear variation is assumed.

**Problem 1.14.** *Derive an expression for the force required for axial movement of a shaft through a taper bearing as shown in figure. The diameter of the shaft is D m and the length is L m. The clearance at the ends are $t_1$m and $t_2$m. The oil has a viscosity of μ and the shaft moves axially at a velocity u.*

In this case the clearance varies along the length and so the velocity gradient will vary along the length. Hence the shear stress also will vary along the length. The total force required can be determined by integrating the elemental force over a differential length *dX*. The clearance, *t* at location *X* is obtained, assuming $t_1 > t_2$,

$$t = t_1 - (t_1 - t_2) \times (X/L) = \{(t_1 \times L) - (t_1 - t_2)\, X\}/L$$

$$du/dy = u/t = u \times L/\{(t_1 \times L) - (t_1 - t_2) \times X\}$$

The velocity gradient at this location is *u/t*, assumed linear.

$$\tau = \mu\,(du/dy),\ dF = \tau\, dA = \tau \times \pi \times D \times dX, \text{ substituting}$$

$$dF = [\{L \times \mu \times u \times \pi \times D\}] \times [dX/\{(t_1 \times L) - (t_1 - t_2) \times X\}]$$

Integrating between the limits $X = 0$ to $X = L$

$$\mathbf{F = [\{\pi \times D \times u \times L \times \mu\}/\{t_1 - t_2\}] \times [\ln(t_1/t_2)]}$$

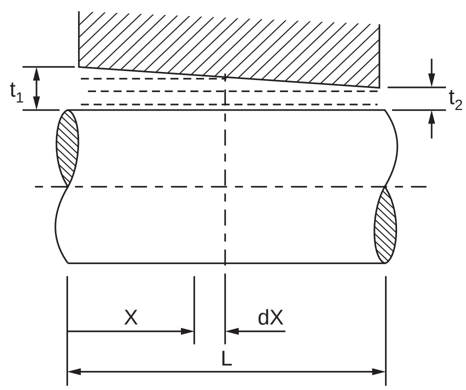



**Figure P.1.14**

**Problem 1.15.** *The clearance between the shaft of 100 mm dia and the bearing varies from 0.2 mm to 0.1 mm over a length of 0.3 m. The viscosity of the oil filling the clearance is $4.8 \times 10^{-2}$ Ns/m². The axial velocity of the shaft is 0.6 m/s.* ***Determine the force required.***

Using the equation derived in the previous problem as given below and substituting the values $F = [\{\pi \times D \times u \times L \times \mu\}/\{t_1 - t_2\}]\ [\ln(t_1/t_2)]$

**F** $= [\{\pi \times 0.1 \times 0.6 \times 0.3 \times 4.8 \times 10^{-2}\}/\{0.0002 - 0.0001\}] \times [\ln(0.0002/0.0001)] =$ **18.814 N**

If the clearance was uniform, $F = \pi \times D \times L \times u \times \mu/t$

For $t = 0.2$ mm, $F_{0.2} = 13.572$ N, For $t = 0.1$ mm, $F_{0.1} = 27.143$ N

The arithmetic average is 20.36 N, while the logarithmic average is what is determined in this problem, 18.814 N.

**Problem 1.16.** ***Derive an expression for the torque required*** *to overcome the viscous resistance when a circular shaft of diameter D rotating at N rpm in a bearing with the clearance t varying uniformly from* $t_1$ *m at one end to* $t_2$*m at the other end. The distance between the ends is L m. The oil has a viscosity of* $\mu$.

In this case the clearance varies along the length and so the velocity gradient ($du/dr$) will vary along the length. Hence the shear stress and the torque also will vary along the length. The total torque required can be determined by integrating the elemental torque over a differential length $dX$.

The clearance, $t$ at location $X$ is obtained, assuming $t_1 > t_2$,

$$t = t_1 - (t_1 - t_2) \times (X/L) = \{(t_1 \times L) - (t_1 - t_2) \times X\}/L$$

The velocity gradient at this location $X$ is $u/t$, as linear profile is assumed.

$$\therefore \quad du/dy = u/t = u \times L/\{(t_1 \times L) - (t_1 - t_2) \times X\}$$

$$\tau = \mu\,(du/dy),\ dF = \tau\,dA = \tau \times \pi \times D \times dX, \text{ substituting}$$

$$dF = [\{L \times \mu \times u \times \pi \times D\}] \times [dX/\{(t_1 \times L) - (t_1 - t_2) \times X\}]$$

Torque = $dF \times (d/2)$ and $u = (\pi DN)/60$. Substituting and Integrating between the limits

$X = 0$ to $X = L$, Torque $\mathbf{= [\{\pi^2 \times D^3 \times L \times N \times \mu\}/\ \{120(t_1 - t_2)\}] \times [\ln (t_1/t_2)]}$

Power = $2\pi NT/60$, hence

$$\mathbf{P = [\{\pi^3 \times D^3 \times L \times N^2 \times \mu\}/\{3600(t_1 - t_2)\}] \times [\ln (t_1/t_2)].}$$

**Problem 1.17** *The clearance between the shaft of 100 mm dia and the bearing varies from 0.2 mm to 0.1 mm over a length of 0.3 m. The viscosity of the oil filling the clearance is* $7.1 \times 10^{-2}$ *Pa.s (Ns/m*$^2$*). The shaft runs at 600 rpm.* ***Determine the torque and power required.***

Using the equations derived in the previous problem as given below and substituting the values

$$T = [\{\pi^2 \times D^3 \times L \times N \times \mu\}/\{120(t_1 - t_2)\}] \times [\ln(t_1/t_2)]$$

$$P = [\{\pi^3 \times D^3 \times L \times N^2 \times \mu\}/\{3600(t_1 - t_2)\}] \times [\ln(t_1/t_2)]$$

$$\mathbf{T} = [\{\pi^2 \times 0.1^3 \times 0.3 \times 600 \times 7.1 \times 10^{-2}\}/\{120(0.0002 - 0.0001)\}] \times [\ln (0.0002/0.0001)]$$

$$= \mathbf{7.29\ Nm.}$$

$$\mathbf{P} = [\{\pi^3 \times 0.1^3 \times 0.3 \times 600^2 \times 7.1 \times 10^{-2}\}/\{3600(0.0002 - 0.0001)\}] \times [\ln (0.0002/0.0001)]$$

$$= \mathbf{457.8\ W.}$$

Check: $P = 2\pi \times 600 \times 7.29/60 = 458\text{W}$.

**Problem 1.18.** ***Determine the capillary depression*** *of mercury in a 4 mm ID glass tube. Assume surface tension as 0.45 N/m and* $\beta = 115°$.

The specific weight of mercury = $13550 \times 9.81$ N/m$^3$, Equating the surface force and the pressure force, $[h \times \gamma \times \pi D^2/4] = [\pi \times D \times \sigma \times \cos\beta]$, Solving for $h$,

$$h = \{4 \times \sigma \times \cos\beta\}/\{\gamma \times D\} = [4 \times 0.45 \times \cos 115]/[13550 \times 9.81 \times 0.004]$$

$$= \mathbf{-1.431 \times 10^{-3}\ m \quad or \quad -1.431\ mm,\ (depression)}$$

**Problem 1.19.** *A ring 200 mm mean dia is to be separated from water surface as shown in figure. The force required at the time of separation was 0.1005 N.* ***Determine the surface tension of water.***

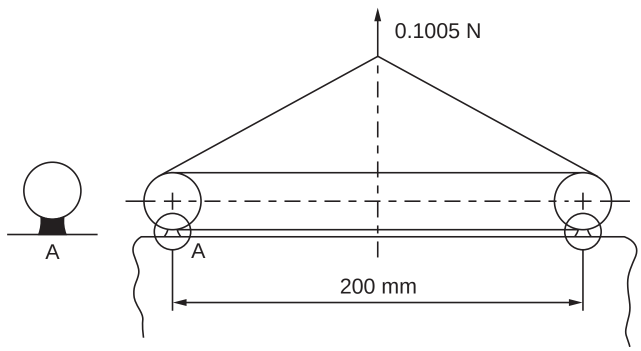



**Figure P.1.19**

The total length of contact just before lifting from the surface will be twice the circumference or $2\pi D$. The force will equal the product of surface tension and the length of contact.

$\sigma \times 2 \times \pi \times 0.2 = 0.1005$ N. Solving $\boldsymbol{\sigma = 0.08}$ **N/m**

The surface tension of a liquid can be measured using this principle provided the fluid wets the surface.

**Problem 1.20.** *A thin plate 1 m wide is slowly lifted vertically from a liquid with a surface tension of 0.1 N/m. Determine what force will be required to overcome the surface tension. Assume* $\beta = 0$.

The total length of contact just before separation from the surface will be twice the width of the plate or $2L$. The force will equal the product of surface tension and the length of contact.

$$\mathbf{F = 2 \times 1 \times 0.1 = 0.02\ N.}$$

**Problem 1.21.** *Diesel injection nozzle sprays fuel with an average diameter of 0.0254 mm. The surface tension is 0.0365 N/m. Determine the pressure difference between the inside and outside of the nozzle. Also determine the pressure difference if the droplet size is reduced to 10 μm.*

A droplet forms at the mouth of the nozzle. The pressure inside the droplet will be higher compared to that at outside.

The equation applicable is $(P_i - P_o) = 2\sigma/R$.

So $(P_i - P_o) = \{2 \times 0.0365 \times 2\}/\{0.0254 \times 10^{-3}\} = \mathbf{5748\ N/m^2 = 5.748\ kN/m^2}$

When the droplet size is reduced to 10 μm the pressure difference is

$$(P_i - P_o) = \{2 \times 0.0365 \times 2\}/\{10 \times 10^{-6}\} = \mathbf{14600\ N/m^2 = 14.6\ kN/m^2.}$$

**Problem 1.22.** *A glass tube of 8 mm ID is immersed in a liquid at 20°C. The specific weight of the liquid is 20601 N/m³. The contact angle is 60°. Surface tension is 0.15 N/m.* ***Calculate the capillary rise*** *and* ***also the radius of curvature of the meniscus****.*

Capillary rise, $h = \{4 \times \sigma \times \cos \beta\}/\{\gamma \times D\} = \{4 \times 0.15 \times \cos 60\}/\{20601 \times 0.008\}$

$= \mathbf{1.82 \times 10^{-3}\ m}$ **or 1.82 mm.**

The meniscus is a doubly curved surface with **equal radius** as the section is circular. (using equation 1.10.3)

$$(P_i - P_o) = \sigma \times \{(1/R_1) + (1/R_2)\} = 2\ \sigma/R$$

$$R = 2\sigma/(P_i - P_o),\ (P_i - P_o) = \text{specific weight} \times h$$

So, $R = [2 \times 0.15]/\ [1.82 \times 10^{-3} \times 2060] = \mathbf{8 \times 10^{-3}\ m}$ **or 8 mm.**

**Problem 1.23.** *A mercury column is used to measure the atmospheric pressure. The height of column above the mercury well surface is 762 mm. The tube is 3 mm in dia. The contact angle is 140°.* ***Determine the true pressure in mm*** *of mercury if surface tension is 0.51 N/m. The space above the column may be considered as vacuum.*

In this case capillary depression is involved and so the true pressure = mercury column + capillary depression.

The specific weight of mercury = 13550 × 9.81 N/m$^3$, equating forces,

$$[h \times \gamma \times \pi D^2/4] = [\pi \times D \times \sigma \times \cos \beta].$$

So $h = \{4 \times \sigma \times \cos \beta\}/\{\gamma \times D\}$

$$h = (4 \times 0.51) \times \cos 140]/[13550 \times 9.81 \times 0.003]$$

$= \mathbf{-\ 3.92 \times 10^{-3}\ m}$ **or – 3.92 mm, (depression)**

Hence actual pressure indicated = 762 + 3.92 = **765.92 mm** of mercury.

**Problem 1.24.** ***Calculate the pressure difference*** *between the inside and outside of a soap bubble of 2.5 mm dia if the surface tension is 0.022 N/m.*

The pressure difference in the case of a sphere is given by, equation 1.10.5

$$(P_i - P_o) = 2\sigma/R = \{2 \times 0.022\}/\{0.0025\} = \mathbf{17.5\ N/m^2.}$$

**Problem 1.25.** *A hollow cylinder of 150 mm OD with its weight equal to the buoyant forces is to be kept floating vertically in a liquid with a surface tension of 0.45 N/m$^2$. The contact angle is 60°. Determine the additional force required due to surface tension.*

In this case a capillary rise will occur and this requires an additional force to keep the cylinder floating.

Capillary rise, $h = \{4 \times \sigma \times \cos \beta\}/\{\gamma \times D\}$.

As $(P_i - P_o) = h \times \text{specific weight},\ (P_i - P_o) = \{4 \times \sigma \times \cos \beta\}/D$

$$(P_i - P_o) = \{4 \times 0.45 \times \cos 60\}/\{0.15\} = 6.0\ \text{N/m}^2$$

$$\text{Force} = \text{Area} \times (P_i - P_o) = \{\pi \times 0.015^2/4\} \times 6 = 0.106\ \text{N}$$

As the immersion leads to additional buoyant force the force required to kept the cylinder floating will be double this value.

So the additional force = 2 × 0.106 = **0.212 N.**

**Problem 1.26.** *The volume of liquid in a rigid piston—cylinder arrangement is 2000 cc. Initially the pressure is 10 bar. The piston diameter is 100 mm. Determine the distance through which the piston has to move so that the pressure will increase to 200 bar. The temperature remains constant. The average value of bulk modulus for the liquid is 2430 × 10$^{-6}$ N/m$^2$.*

By definition—refer eqn 1.11.1

$$E_v = -\,dP/(dv/v) = -\,(P_2 - P_1)/[(v_2 - v_1)/v_1]$$

So $\quad 2430 \times 10^6 = -190 \times 10^5/(dv/0.002)$, Solving,

$$dv = -\,0.002 \times 190 \times 10^5/2430 \times 10^6 = 15.64 \times 10^{-6}\ \text{m}^3$$

Piston movement, $L = dv/\text{area}$

$$L = dv \times 4/\pi D^2 = 15.64 \times 10^{-6} \times 4/\pi \times 0.1^2 = 1.991 \times 10^{-3}\ \text{m} = \mathbf{1.991\ mm}$$

(the piston-cylinder arrangement is assumed to be rigid so that there is no expansion of the container)

**Problem 1.27.** *The pressure of water increases with depth in the ocean. At the surface, the density was measured as 1015 kg/m*$^3$*. The atmospheric pressure is 1.01 bar. At a certain depth, the pressure is 880 bar.* ***Determine the density of sea water at the depth.*** *The average value of bulk modulus is 2330 ×* $10^6$ *N/m*$^2$*.*

The density will increase due to the pressure increase.

Bulk modulus is defined in eqn 1.11.1 as $E_v = -\,dP/(dv/v) = -\,(P_2 - P_1)/[v_2 - v_1)/v_1]$,

$$[(v_2 - v_1)/v_1] = -\,(P_2 - P_1)/E_v = -\,[880 \times 10^5 - 1.01 \times 10^5]/2330 \times 10^6 = -0.03772$$

$v_1 = 1/1015\ \text{m}^3/\text{kg}$, substituting the values in

$v_2 = [v_1 \times \{-\,(P_2 - P_1)/E_v\}] + v_1,$

$v_2 = [-\,0.03772 \times (1/1015)] + (1/1015) = 9.48059 \times 10^{-4}\ \text{m}^3/\text{kg}$

**Density** = $1/(9.48059 \times 10^{-4}\ \text{m}^3/\text{kg})$ = **1054.79 kg/m$^3$** an increase of 4%.

**The density increases by 4.0% due to the increase in pressure.**

$[(v_2 - v_1)/v_1]$ also equals $[(\rho_1 - \rho_2)/\rho_2] = [(P_2 - P_1)/E_v]$

Use of this equation should also give the same answer.

**Problem 1.28.** *A diesel fuel pump of 10 mm ID is to deliver against a pressure of 200 bar. The fuel volume in the barrel at the time of closure is 1.5 cc. Assuming rigid barrel determine the plunger movement before delivery begins. The bulk modulus of the fuel is 1100 ×* $10^6$ *N/m*$^2$*.*

By definition—eqn 1.11.1—the bulk modulus is $E_v = -\,dP/(dv/v)$,

$$1100 \times 10^6 = -\,200 \times 10^5/(dv\,/\,1.5 \times 10^{-6}),\ \text{Solving}\ dv = -\,2.77 \times 10^{-8}\ \text{m}^3$$

Plunger movement = $dv/\text{area} = -\,2.77 \times 10^{-8} \times 4/(\pi \times 0.0015^2)$

$$= 3.47 \times 10^{-4}\ \text{m} = \mathbf{0.347\ mm}$$

(the pressure rise will also be affected by the expansion of the pipe line).

# *Chapter-2* ***Bernoulli Equation and Applications***

## 2.0 INTRODUCTION

In chapter five flow of ideal fluids was discussed. The main idea was the study of flow pattern. The determination of equal flow paths and equal potential lines was discussed. No attempt was made to determine the numerical value of these quantities.

In this chapter the method of determination of the various energy levels at different locations in the flow is discussed. In this process first the various forms of energy in the fluid are identified. Applying the law of conservation of energy the velocity, pressure and potential at various locations in the flow are calculated. Initially the study is limited to ideal flow. However the modifications required to apply the analysis to real fluid flows are identified.

The material discussed in this chapter are applicable to many real life fluid flow problems. The laws presented are the basis for the design of fluid flow systems.

**Energy consideration in fluid flow:**

Consider a small element of fluid in flow field. The energy in the element as it moves in the flow field is conserved. This principle of conservation of energy is used in the determination of flow parameters like pressure, velocity and potential energy at various locations in a flow. The concept is used in the analysis of flow of ideal as well as real fluids.

Energy can neither be created nor destroyed. It is possible that one form of energy is converted to another form. The total energy of a fluid element is thus conserved under usual flow conditions.

If a stream line is considered, it can be stated that the total energy of a fluid element at any location on the stream line has the same magnitude.

## 2.1 FORMS OF ENERGY ENCOUNTERED IN FLUID FLOW

Energy associated with a fluid element may exist in several forms. These are listed here and the method of calculation of their numerical values is also indicated.

### 2.1.1 Kinetic Energy

This is the energy due to the motion of the element as a whole. If the velocity is $V$, then the kinetic energy for m kg is given by

$$KE = \frac{mV^2}{2g_o} \text{ Nm} \tag{2.1.1}$$

The unit in the SI system will be Nm also called Joule ($J$)

{(kg $m^2/s^2$)/(kg m/N $s^2$)}

The same referred to one kg (specific kinetic energy) can be obtained by dividing 2.1.1 by the mass m and then the unit will be Nm/kg.

$$KE = \frac{V^2}{2g_o}, \text{ Nm/kg} \tag{2.1.1b}$$

In fluid flow studies, it is found desirable to express the energy as the head of fluid in m. This unit can be obtained by multiplying equation (6.1.1) by $g_o/g$.

$$\text{Kinetic head} = \frac{V^2}{2g_o}\frac{g_o}{g} = \frac{V^2}{2g}$$

The unit for this expression will be $\frac{m^2 s^2}{s^2 m}$ = m

Apparently the unit appears as metre, but in reality it is Nm/N, where the denominator is weight of the fluid in N.

The equation in this form is used at several places particularly in flow of liquids. But the energy associated physically is given directly only be equation 6.1.1.

The learner should be familiar with both forms of the equation and should be able to choose and use the proper equation as the situation demands. **When different forms of the energy of a fluid element is summed up to obtain the total energy, all forms should be in the same unit.**

### 2.1.2 Potential Energy

This energy is due to the position of the element in the gravitational field. While a zero value for $KE$ is possible, the value of potential energy is relative to a chosen datum. The value of potential energy is given by

$$PE = mZ\, g/g_o \text{ Nm} \tag{2.1.3}$$

Where $m$ is the mass of the element in kg, Z is the distance from the datum along the gravitational direction, in $m$. The unit will be (kg m m/$s^2$) × ($Ns^2$/kgm) *i.e.,* Nm. The specific potential energy (per kg) is obtained by dividing equation 6.1.3 by the mass of the element.

$$PE = Z\, g/g_0 \text{ Nm/kg} \tag{2.1.3. b}$$

This gives the physical quantity of energy associated with 1 kg due to the position of the fluid element in the gravitational field above the datum. As in the case of the kinetic energy, the value of $PE$ also is expressed as head of fluid, $Z$.

$$PE = Z\,(g/g_o)\,(g_o/g) = Z\, m. \tag{2.1.4}$$

This form will be used in equations, but as in the case of KE, one should be familiar with both the forms and choose the suitable form as the situation demands.

### 2.1.3 Pressure Energy (Also Equals Flow Energy)

The element when entering the control volume has to flow against the pressure at that location. The work done can be calculated referring Fig. 2.1.1.

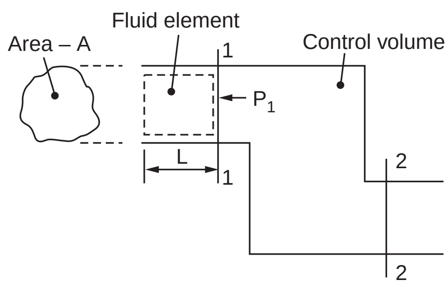



**Figure 2.1.1** Flow work calculation

The boundary of the element of fluid considered is shown by the dotted line, Force = $P_1 A$, distance to be moved = $L$, work done = $P_1AL$ = $P_1\ mv$ as $AL$ = volume = mass × specific volume, $v$. $\therefore$ flow work = $P\ mv$.

The pressure energy per kg can be calculated using $m$ = 1. The flow energy is given by

$$FE = P.v = P/\rho, \text{ Nm/kg} \tag{2.1.5}$$

Note: $$\frac{N}{m^2}\frac{m^3}{kg} \rightarrow \frac{Nm}{kg}$$

As in the other cases, the flow energy can also expressed as head of fluid.

$$FE = \frac{P}{\rho}\frac{g_o}{g}, \text{ m} \tag{2.1.5a}$$

As specific weight $\gamma = \rho\ g/g_o$, the equation is written as,

$$FE = P/\gamma, \text{ m} \tag{2.1.5b}$$

It is important that in any equation, when energy quantities are summed up consistent forms of these set of equations should be used, that is, all the terms should be expressed either as head of fluid or as energy (J) per kg. These are the three forms of energy encountered more often in flow of incompressible fluids.

### 2.1.4 Internal Energy

This is due to the thermal condition of the fluid. This form is encountered in compressible fluid flow. For gases (above a datum temperature) $IE = c_v\ T$ where $T$ is the temperature above the datum temperature and $c_v$ is the specific heat of the gas at constant volume. The unit for internal energy is J/kg (Nm/kg). When friction is significant other forms of energy is converted to internal energy both in the case of compressible and incompressible flow.

### 2.1.5 Electrical and Magnetic Energy

These are not generally met with in the study of flow of fluids. However in magnetic pumps and in magneto hydrodynamic generators where plasma flow in encountered, electrical and magnetic energy should also be taken into account.

## 2.2 VARIATION IN THE RELATIVE VALUES OF VARIOUS FORMS OF ENERGY DURING FLOW

Under ideal conditions of flow, if one observes the movement of a fluid element along a stream line, the sum of these forms of energy will be found to remain constant. However, there may be an increase or decrease of one form of energy while the energy in the other forms will decrease or increase by the same amount. For example when the level of the fluid decreases, it is possible that the kinetic energy increases. When a liquid from a tank flows through a tap this is what happens. In a diffuser, the velocity of fluid will decrease but the pressure will increase. In a venturimeter, the pressure at the minimum area of cross section (throat) will be the lowest while the velocity at this section will be the highest.

The total energy of the element will however remain constant. In case friction is present, a part of the energy will be converted to internal energy which should cause an increase in temperature. But the fraction is usually small and the resulting temperature change will be so small that it will be difficult for measurement. From the measurement of the other forms, it will be possible to estimate the frictional loss by difference.

## 2.3 EULER'S EQUATION OF MOTION FOR FLOW ALONG A STREAM LINE

Consider a small element along the stream line, the direction being designated as $s$.

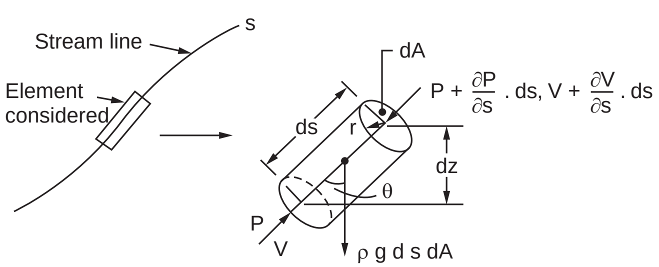



**Figure 2.3.1** Euler's equation of Motion – Derivation

The net force on the element are the body forces and surface forces (pressure). These are indicated in the figure. Summing this up, and equating to the change in momentum.

$$PdA - \{P + (\partial P/\partial s\} \, dA - \rho g \, dA \, ds \cos\theta = \rho \, dA \, ds \, a_s \qquad (2.3.1)$$

where $a_s$ is the acceleration along the $s$ direction. This reduces to,

$$\frac{1}{\rho}\frac{\partial P}{\partial s} + g\cos\theta + a_s = 0 \qquad (2.3.2)$$

(**Note**: It will be desirable to add $g_o$ to the first term for dimensional homogeneity. As it is, the first term will have a unit of N/kg while the other two terms will have a unit of m/s$^2$. Multiplying by $g_o$, it will also have a unit of m/s$^2$).

$$a_s = dV/dt, \text{ as velocity, } V = f(s, t), (t = \text{time}).$$

$$dV = \frac{\partial V}{\partial s} ds + \frac{\partial V}{\partial t} dt \quad \text{dividing by } dt,$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial s}\frac{ds}{dt} + \frac{\partial V}{\partial t} \qquad \text{As } \frac{ds}{dt} = V,$$

and as cos θ = $dz/ds$, equation 6.3.2 reduces to,

$$\frac{1}{\rho}\frac{\partial P}{\partial s} + g\frac{\partial z}{\partial s} + V\frac{\partial V}{\partial s} + \frac{\partial V}{\partial t} = 0 \tag{2.3.2. a}$$

For steady flow $\partial V/\partial t = 0$. Cancelling $\partial s$ and using total derivatives in place of partials as these are independent quantities.

$$\frac{dp}{\rho} + gdz + VdV = 0 \tag{2.3.3}$$

(**Note**: in equation 2.3.3 also it is better to write the first term as $g_o.dp/\rho$ for dimensional homogeneity).

This equation after dividing by $g$, is also written as,

$$\frac{dp}{\gamma} + d\left(\frac{V^2}{2g}\right) + dz = 0 \quad \text{or} \quad d\left[\frac{P}{\gamma} + \frac{V^2}{2g} + z\right] = 0 \tag{2.3.4}$$

which means that the quantity within the bracket remains constant along the flow.

This equation is known as Euler's equation of motion. The assumptions involved are:

1. Steady flow
2. Motion along a stream line and
3. Ideal fluid (frictionless)

In the case on incompressible flow, this equation can be integrated to obtain Bernoulli equation.

## 2.4 BERNOULLI EQUATION FOR FLUID FLOW

Euler's equation as given in 2.3.3 can be integrated directly if the flow is assumed to be incompressible.

$$\frac{dP}{\rho} + gdz + VdV = 0, \quad \text{as } \rho = \text{constant}$$

$$\frac{P}{\rho} + gz + \frac{V^2}{2} = \text{const. or } \frac{P}{\rho} + z\left(\frac{g}{g_0}\right) + \frac{V^2}{2g_0} = \text{Constant} \tag{2.4.1}$$

The constant is to be evaluated by using specified boundary conditions. The unit of the terms will be energy unit (Nm/kg).

In SI units the numerical value of $g_o$ = 1, kg m/N s$^2$. Equation 6.4.1 can also be written as to express energy as head of fluid column.

$$\frac{P}{\gamma} + z + \frac{V^2}{2g} = \text{constant} \tag{2.4.2}$$

($\gamma$ is the specific weight N/m$^3$). In this equation all the terms are in the unit of head of the fluid.

The constant has the same value along a stream line or a stream tube. The first term represents (flow work) pressure energy, the second term the potential energy and the third term the kinetic energy.

This equation is extensively used in practical design to estimate pressure/velocity in flow through ducts, venturimeter, nozzle meter, orifice meter etc. In case energy is added or taken out at any point in the flow, or loss of head due to friction occurs, the equations will read as,

$$\frac{P_1}{\rho} + \frac{V_1^2}{2g_o} + \frac{z_1 g}{g_o} + W - \frac{h_f g}{g_o} = \frac{P_2}{\rho} + \frac{V_2^2}{2g_o} + \frac{z_2 g}{g_o}$$

where $W$ is the energy added and $h_f$ is the loss of head due to friction.

In calculations using SI system of units $g_o$ may be omitting as its value is unity.

**Example 2.1** *A liquid of specific gravity 1.3 flows in a pipe at a rate of 800 l/s, from point 1 to point 2 which is 1 m above point 1. The diameters at section 1 and 2 are 0.6 m and 0.3 m respectively. If the pressure at section 1 is 10 bar,* ***determine the pressure at section 2.***

Using Bernoulli equation in the following form (2.4.2)

$$\frac{P}{\gamma} + z + \frac{V^2}{2g} = \text{constant},$$

Taking the datum as section 1, the pressure $P_2$ can be calculated.

$V_1 = 0.8 \times 4/\pi \times 0.6^2 = 2.83$ *m/s*, $V_2 = 0.8 \times 4/\pi \times 0.3^3 = 11.32$ *m/s*

$P_1 = 10 \times 10^5$ N/m$^2$, $\gamma$ = sp. gravity × 9810. Substituting.

$$\frac{10 \times 10^5}{9810 \times 1.3} + 0 + \frac{2.83^2}{2 \times 9.81} = \frac{\text{P}_2}{9810 \times 1.3} + 1 + \frac{11.32^2}{2 \times 9.81}$$

Solving, $\mathbf{P_2 = 9.092}$ **bar** ($9.092 \times 10^5$ N/m$^2$).

As $P/\gamma$ is involved directly on both sides, gauge pressure or absolute pressure can be used without error. However, it is desirable to use absolute pressure to avoid negative pressure values (or use of the term vacuum pressure).

**Example 2.2** *Water flows through a horizontal venturimeter with diameters of 0.6 m and 0.2 m. The guage pressure at the entry is 1 bar.* ***Determine the flow rate*** *when the throat pressure is 0.5 bar (vacuum). Barometric pressure is 1 bar.*

Using Bernoulli's equation in the form,

$$\frac{P_1}{\gamma} + Z_1 + \frac{V_1^2}{2g} = \frac{P_2}{\gamma} + Z_2 + \frac{V_2^2}{2g}$$

and noting $Z_1 = Z_2$, $P_1 = 2 \times 10^5$ N/m$^2$ (absolute)

$P_2 = 0.5 \times 10^5$ N/m$^2$ (absolute), $\gamma = 9810$ N/m$^3$

$V_1 = Q \times 4/(\pi \times 0.60^2) = 3.54\,Q$, $V_2 = Q \times 4/(\pi \times 0.20^2) = 31.83Q$

$$\frac{2 \times 10^5}{9810} + 0 + \frac{3.54^2}{2 \times 9.81} Q^2 = \frac{0.5 \times 10^5}{9810} + 0 + \frac{31.83^2\, Q^2}{2 \times 9.81}$$

Solving, **Q = 0.548 m³/s**, $V_1 = 1.94$ m/s, $V_2 = 17.43$ m/s.

**Example 2.3** *A tap discharges water evenly in a jet at a velocity of 2.6 m/s at the tap outlet, the diameter of the jet at this point being 15 mm. The jet flows down vertically in a smooth stream.* ***Determine the velocity and the diameter*** *of the jet at 0.6 m below the tap outlet.*

The pressure around the jet is atmospheric throughout. Taking the tap outlet as point 1 and also taking it as the datum using Bernoulli equation.

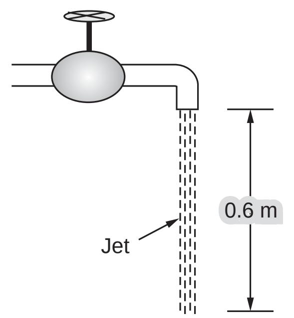



**Figure Ex. 2.3** Problem model

$$\frac{P_1}{\gamma} + Z_1 + \frac{V_1^2}{2g} = \frac{P_2}{\gamma} + Z_2 + \frac{V_2^2}{2g},$$

$P_1 = P_2$, $Z_2 = 0$,

$Z_2 = -0.6$ m, $V_1 = 2.6$ m/s

$$\therefore \quad \frac{2.6^2}{2 \times 9.81} = -0.6 + \frac{V_2^2}{2 \times 9.81}$$

$\therefore$ **$V_2$ = 4.3 m/s.**

using continuity equation (one dimensional flow) and noting that density is constant.

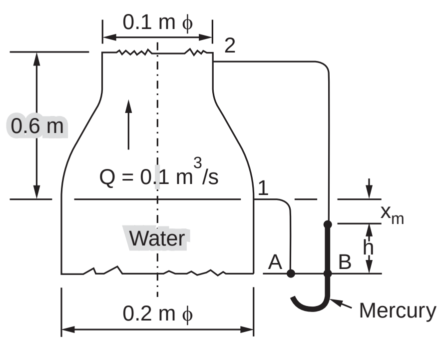
$$A_1V_1 = A_2V_2$$

$$\frac{\pi \times 015^2}{4} \times 2.6 = \frac{\pi \times D^2}{4} \times 4.3, \quad \therefore \quad \textbf{D = 0.01166 m or 11.66 mm}$$

As the potential energy decreases, kinetic energy increases. As the velocity is higher the flow area is smaller.

Entrainment of air may increase the diameter somewhat.

**Example 2.4** *Water flows in a tapering pipe vertically as shown in Fig. Ex.6.4.* ***Determine the manometer reading "h"****. The manometer fluid has a specific gravity of 13.6. The flow rate is 100 l/s*



The velocities at sections 1 and 2 are first calculated.

$$V_1 = 4 \times 0.1/(\pi \times 0.2^2) = 3.183 \text{ m/s},$$

$$V_2 = 4 \times 0.1/(\pi \times 0.1^2) = 12.732 \text{ m/s}$$

It is desired to determine $P_1 - P_2$. Rearranging Bernoulli equation for this flow,

$$\frac{P_1 - P_2}{\gamma} = 0.6 + (12.732^2 - 3.183^2)/(2 \times 9.81) = 8.346 \text{ m of water}$$

For water $\gamma$ = 9810 N/m$^3$. For the manometer configuration, considering the level $AB$ and equating the pressures at $A$ and $B$

$$\frac{P_1}{\gamma} + x + h = \frac{P_2}{\gamma} + 0.6 + x + sh$$

(where $x$, $h$ are shown on the diagram and $s$ is specific gravity)

$\therefore$ $$\frac{P_1 - P_2}{\gamma} = 0.6 + h(s - 1), \text{ substituting the values,}$$

$$8.346 = 0.6 + h(13.6 - 1)$$

$\therefore$ **h = 0.6148 m or 61.48 cm**

## 2.5 ENERGY LINE AND HYDRAULIC GRADIENT LINE

The total energy plotted along the flow to some specified scale gives the energy line. When losses (frictional) are negligible, the energy line will be horizontal or parallel to the flow direction. For calculating the total energy kinetic, potential and flow (pressure) energy are considered.

Energy line is the plot of $\frac{P}{\gamma} + Z + \frac{V^2}{2g}$ along the flow. It is constant along the flow when losses are negligible.

The plot of $\frac{P}{\gamma} + Z$ along the flow is called the hydraulic gradient line. When velocity increases this will dip and when velocity decreases this will rise. An example of plot of these lines for flow from a tank through a venturimeter is shown in Fig. 6.5.1.

The hydraulic gradient line provides useful information about pressure variations (static head) in a flow. The difference between the energy line and hydraulic gradient line gives the value of dynamic head (velocity head).

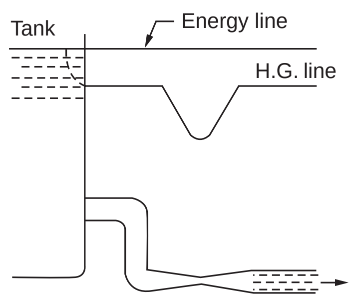



**Figure 2.5.1** Energy and hydraulic gradient lines

## 2.6 VOLUME FLOW THROUGH A VENTURIMETER

**Example 2.6** *Under ideal conditions show that the volume flow through a venturimeter is given by*

$$Q = \frac{A_2}{\left\{1-(A_2/A_1)^2\right\}^{0.5}}\left[2g\left(\frac{P_1-P_2}{\gamma}+(Z_1-Z_2)\right)\right]^{0.5}$$

*where suffix 1 and 2 refer to the inlet and the throat.*

Refer to Fig. Ex. 6.5

Volume flow $= A_1 V_1 = A_2 V_2$

$$\therefore \quad V_1 = \frac{A_2}{A_1} V_2,\ V_1^2 = \left(\frac{A_2}{A_1}\right)^2 . V_2^2,$$

$$\therefore \quad (V_2^2 - V_1^2) = V_2^2\left[1-\left(\frac{A_2}{A_1}\right)^2\right]$$

Applying Bernoulli equation to the flow and considering section 1 and 2,

$$\frac{P_1}{\gamma} + Z_1 + \frac{V_1^2}{2g} = \frac{P_2}{\gamma} + Z_2 + \frac{V_2^2}{2g}$$

Rearranging,

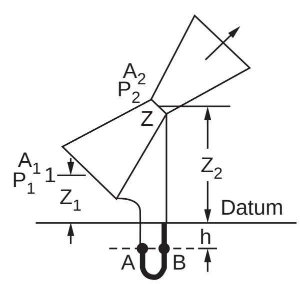
$$\left[2g\left\{\frac{P_1-P_2}{\gamma}+(Z_1-Z_2)\right\}\right]^{0.5} = V_2\left[1-\left(\frac{A_2}{A_1}\right)^2\right]^{0.5}$$

$$V_2 = \frac{1}{[1-(A_2/A_1)^2]^{0.5}}\left[2g\left\{\frac{P_1-P_2}{\gamma}+(Z_1-Z_2)\right\}\right]^{0.5}$$

$\therefore$ Volume flow is

$$A_2V_2 = \frac{A_2}{[1-(A_2/A_1)^2]^{0.5}}\left[2g\left\{\frac{P_1-P_2}{\gamma}+(Z_1-Z_2)\right\}\right]^{0.5}$$

This is a general expression and can be used irrespective of the flow direction, inclination from horizontal or vertical position. This equation is applicable for orifice meters and nozzle flow meters also.




In numerical work consistent units should be used.

Pressure should be in N/m$^2$, $Z$ in m, $A$ in m$^2$ and then volume flow will be m$^3$/s.

A coeficient is involved in actual meters due to friction.

**Example 6.6** *Show that when a manometric fluid of specific gravity $S_2$ is used to measure the head in a venturimeter with flow of fluid of specific gravity $S_1$, if the manometer shows a reading of hm, the volume flow is given by*

$$Q = \frac{A_2}{\left[1-(A_2/A_1)^2\right]^{0.5}}\left[2gh\left(\frac{S_2}{S_1}-1\right)\right]^{0.5}$$

Comparing the equation (6.6.1) with the problem at hand, it is seen that it is sufficient to prove,

$$h\left(\frac{S_2}{S_1}-1\right) = \frac{P_1-P_2}{\gamma_1} + (Z_2 - Z_1)$$

Considering the plane *A–B* in the manometer and equating the pressures at *A* and *B* Fig. Ex. 6.5 : The manometer connection at the wall measures the static pressure only)

$$P_1 + Z_1\,\gamma_1 + h\gamma_1 = P_2 + Z_2\gamma_1 + h\gamma_2$$

$$(P_1 - P_2) + (Z_1 - Z_1)\,\gamma_1 = h(\gamma_2 - \gamma_1),\ \text{dividing by } \gamma_1,$$

$$\frac{P_1-P_2}{\gamma_1} + (Z_1 - Z_2) = h\left(\frac{\gamma_2}{\gamma_1}-1\right) = h\left(\frac{S_2}{S_1}-1\right)$$

Hence volume flow,

$$Q = \frac{A_2}{[1-(A_2/A_1)^2]^{0.5}}\left[2gh\left(\frac{S_2}{S_1}-1\right)\right]^{0.5}$$

This equation leads to another conclusion. The fluid head, $H$, causing the flow is equal to the manometer reading $h[(S_2/S_1) - 1]$ and flow is independent of the inclination if the reading of the manometer and the fluids are specified.

*i.e.,* As the manometer reading converted to head of flowing fluid, $H = h[(S_2/S_1) - 1]$

$$Q = \frac{A_2}{[1-(A_2/A_1)^2]^{0.5}}[2gH]^{0.5}$$

If the pressure at various locations are specified, these equations are applicable for orifice and nozzle meters also.

**Example. 2.7** ***Determine the flow rate*** *through the siphon Fig. Ex. 6.7 when flow is established. Also determine the pressure at A.*

The pressure at $C$ and $B$ are atmospheric. Considering locations $C$ and $B$ and taking the datum at $B$, applying Bernoulli equation, noting that the velocity at water surface at $C = 0$.

$$0 + 0 + V_B^2/2g = 3 + 0 + 0$$

$$\therefore \quad V_B = 7.672 \text{ m/s.}$$

$$\therefore \quad \textbf{Flow rate} = (\pi D^2/4) \times V$$

$$= (\pi \times 0.1^2/4) \times 7.672$$

$$= \mathbf{0.06\ m^3/s}$$

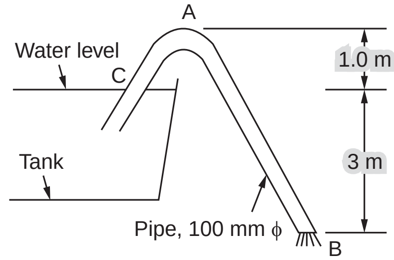



**Figure Ex. 2.7** Problem model

The velocity at $A$ is the same as velocity at $B$. Now considering locations $C$ and $A$,

$$3 + 0 + 0 = 4 + (P_A/\gamma) + 7.672^2/(2 \times 9.81)$$

$\therefore$ $P_A/\gamma = -4$m or $-4$m of water head or 4m water-head below atmospheric pressure.

**Check**: Consider points $A$ and $B$

$$4 + \frac{P_A}{\gamma} + \frac{V_A^2}{2g} = \frac{V_B^2}{2g} + 0 + 0 \quad \text{as } V_A = V_B, \quad \frac{\mathbf{P_A}}{\gamma} = \mathbf{-4\ m} \text{ checks.}$$

**Example. 2.8** *Water flows in at a rate of 80 l/s from the pipe as shown in Fig. Ex.2 .8 and flows outwards through the space between the top and bottom plates. The top plate is fixed.* ***Determine the net force acting on the bottom plate****. Assume the pressure at radius r = 0.05 m is atmospheric*.

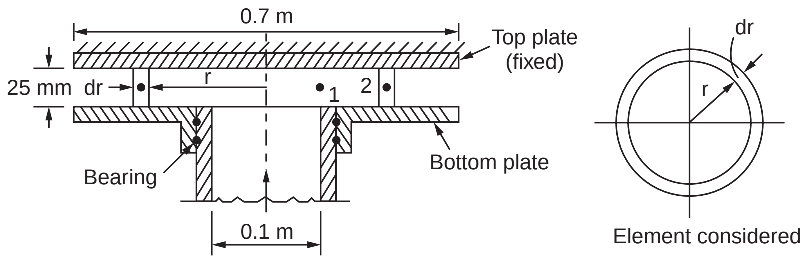



**Figure Ex. 2.8** Problem model

Consider an element area of width $dr$ (annular) in the flow region at a distance $r$ as shown in figure. The pressure at this location as compared to point 1 can be determined using Bernoulli equation.

$$\frac{P_1}{\gamma} + Z_1 + \frac{V_1^2}{2g} = \frac{P_2}{\gamma} + Z_2 + \frac{V_2^2}{2g}, \; P_1 \text{ is atmospheric}$$

As $Z_1 = Z_2$,

$$P_2 - P_1 = \frac{\gamma}{2g}(V_1^2 - V_2^2)$$

$$V_1^2 = (0.08/2\pi \times 0.05 \times 0.025)^2 = 103.75$$

$$V_2^2 = (0.08/2\pi \times 0.025 \times r)^2 = 0.2594/r^2$$

$(P_2 - P_1)$ is the pressure difference which causes a force at the area $2\pi r dr$ at $r$.

The force on the element area of the bottom plate = $2\pi r dr\,(P_2 - P_1)$

Substituting and nothing $\gamma = \rho g/g_0$, the elemental force $dF$ is given by,

$$dF = \rho\, \pi r dr \left[103.75 - \frac{0.2594}{r^2}\right],$$

Integrating between the limits $r = 0.05$ to 0.35,

$$\textbf{Net force} = 1000 \times \pi \left[(103.75\,(0.35^2 - 0.05^2)/2) - \left(0.2594 \ln \frac{0.35}{0.05}\right)\right] = \mathbf{17970\ \ N}$$

## 2.7 EULER AND BERNOULLI EQUATION FOR FLOW WITH FRICTION

Compared to ideal flow the additional force that will be involved will be the shear force acting on the surface of the element. Let the shear stress be $\tau$, the force will equal $\tau\, 2\pi r\, ds$ (where $r$ is the radius of the element, and $A = \pi\, r^2$)

Refer **Para 2.3** and **Fig. 2.3.1.** The Euler equation 6.3.3 will now read as

$$\frac{dP}{\rho} + VdV + gdZ - \frac{2\tau ds}{\rho r} = 0$$

$$\frac{dP}{\gamma} + d\left(\frac{V^2}{2g}\right) + dZ - \frac{2\tau ds}{\gamma\, r} = 0$$

$ds$ can also be substituted in terms of $Z$ and $\theta$

Bernoulli equation will now read as (taking $s$ as the length)

$$\frac{P_1}{\gamma} + \frac{V_1^{\,2}}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^{\,2}}{2g} + Z_2 + \frac{2\tau s}{\gamma r}$$

The last term is the loss of head due to friction and is denoted often as $h_L, h_f$ in head of fluid in metre height (check for the unit of the last term).

**Example 2.9** *The delivery line of a pump is 100 mm ID and it delivers water at a height of 12 m above entry. The pipe ends in a nozzle of diameter 60 mm. The total head at the entry to the pipe is 20 m.* ***Determine the flow rate if losses in the pipe is given by 10 $V_2^2/2g$****. where $V_2$ is the velocity at nozzle outlet. There is no loss in the nozzle.*

Equating the total energy at inlet and outlet,

$$20 = 12 + \frac{V_2^2}{2g} + 10\,\frac{V_2^2}{2g},$$

$$\therefore \qquad V_2^{\,2} = \frac{8 \times 2 \times 9.81}{11}, \; V_2 = 3.777 \text{ m/s}$$

$$\textbf{Flow} = A_2\,V_2 = \frac{\pi \times 0.06^2}{4} \times 3.777 = 0.01068 \text{ m}^3\text{/s} = \mathbf{0.64\ m^3/min.}$$

(If losses do not occur then, $V_2 = 12.53$ m/s and flow will be 2.13 $m^3$/min)

**Example 2.10** *A tank with water level of 12 m has a pipe of 200 mm dia connected from its bottom which extends over a length to a level of 2 m below the tank bottom.* ***Calculate the pressure at this point*** *if the flow rate is 0.178 $m^3$/s. The losses due to friction in the pipe length is accounted for by 4.5 $V_2^2/2g$.*

Taking location of the outlet of the pipe as the datum, using Bernoulli equation and accounting for frictional drop in head (leaving out the atmospheric pressure which is the same at the water level and at outlet).

$$14 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + 4.5\,\frac{V_2^2}{2g}$$

$\therefore$ $$V_2 = 0.178/\pi \times 0.1 \times 0.1 = 5.67 \text{ m/s}$$

$$14 - 5.5 \times \frac{5.67^2}{2 \times 9.81} = \frac{P_2}{\gamma} = 5 \text{ m of water head.}$$

$\therefore$ $P_2 = 9810 \times 5$ N/m$^2$ = **0.49 bar** (above atmospheric pressure)

**Example 2.11** *A vertical pipe of diameter of 30 cm carrying water is reduced to a diameter of 15 cm. The transition piece length is 6 m. The pressure at the bottom is 200 kPa and at the top it is 80 kPa. If frictional drop is 2 m of water head,* ***determine the rate*** *of flow.*

Considering the bottom as the datum,

$$\frac{200 \times 10^3}{9810} + 0 + \frac{V_1^2}{2g} = \frac{80 \times 10^3}{9810} + 6 + \frac{V_2^2}{2g} + 2$$

$$V_2^2 = V_1^2\,(0.3/0.15)^4 = 16V_1^2$$

$\therefore$ $$\frac{120 \times 10^3}{9810} - 8 = 15\frac{V_1^2}{2g}, \text{ Solving, } V_1 = 2.353 \text{ and } V_2 = 9.411 \text{ m/s}$$

$\therefore$ **Flow rate** = $A_1 V_1 = A_2 V_2$ = **0.166 m$^3$/s**

## 2.8 CONCEPT AND MEASUREMENT OF DYNAMIC, STATIC AND TOTAL HEAD

In the Bernoulli equation, the pressure term is known as static head. It is to be measured by a probe which will be perpendicular to the velocity direction. Such a probe is called static probe. The head measured is also called Piezometric head. (Figure 2.8.1 (*a*))

The velocity term in the Bernoulli equation is known as dynamic head. It is measured by a probe, one end of which should face the velocity direction and connected to one limb of a manometer with other end perpendicular to the velocity and connected to the other limb of the manometer. (Figure 2.8.1 (*b*))

The total head is the sum of the static and dynamic head and is measured by a single probe facing the flow direction. (Figure 2.8.1 (*c*))

The location of probes and values of pressures for the above measurements are shown in Fig.2.8.1.

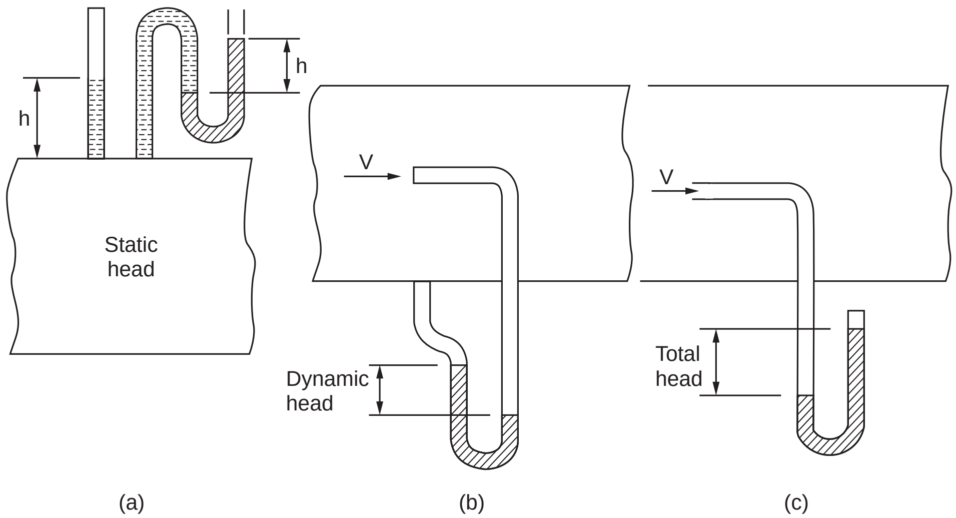



**Figure 2.8.1** Pressure measurement

## 2.8.1 Pitot Tube

The flow velocity can be determined by measuring the dynamic head using a device known as pitot static tube as shown in Fig. 2.8.2. The holes on the outer wall of the probe provides the static pressure (perpendicular to flow) and hole in the tube tip facing the stream direction of flow measures the total pressure. The difference gives the dynamic pressure as indicated by the manometer. The head will be $h$ $(s - 1)$ of water when a differential manometer is used $(s > 1)$.

The velocity variation along the radius in a duct can be conveniently measured by this arrangement by traversing the probe across the section. This instrument is also called pitot–static tube.

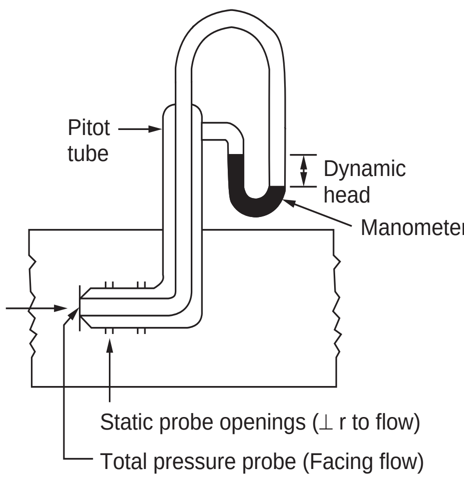



**Figure 2.8.2** Pitot-Static tube

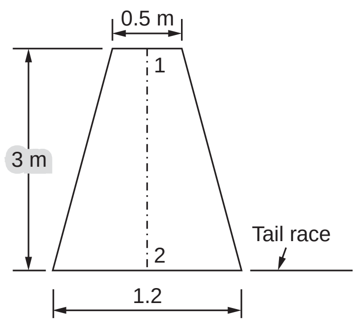
**Example 2.12** *The dynamic head of a water jet stream is measured as 0.9 m of mercury column.* ***Determine the height to which the jet will rise*** *when it is directed vertically upwards.*

Considering the location at which the dynamic head is measured as the datum and converting the column of mercury into head of water, and noting that at the maximum point the velocity is zero,

$$0.9 \times 13.6 + 0 + 0 = 0 + 0 + Z \quad \therefore \mathbf{Z = 12.24\ m}$$

**Note.** If the head measured is given as the reading of a differential manometer, then the head should be calculated as 0.9 (13.6 – 1) m.

**Example 2.13** *A diverging tube connected to the outlet of a reaction turbine (fully flowing) is called "Draft tube". The diverging section is immersed in the tail race water and this provides additional head for the turbine by providing a pressure lower than the atmospheric pressure at the turbine exit. If the turbine outlet is open the exit pressure will be atmospheric as in Pelton wheel. In a draft tube as shown in Fig. Ex. 6.13,* ***calculate the additional head provided by the draft tube.*** *The inlet diameter is 0.5 m and the flow velocity is 8 m/s. The outlet diameter is 1.2 m. The height of the inlet above the water level is 3 m. Also calculate the pressure at the inlet section.*




Considering sections 1 and 2

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} Z_1 = \frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_2$$

Considering tail race level, 2 as the datum, and calulating the velocities

$$V_1 = 8 \text{ m/s}, V_2 = 8 \times \frac{0.5^2}{1.2^2} = 1.39 \text{ m/s.}$$

$$P_2 = \text{atmospheric pressure}, Z_2 = 0, Z_1 = 3$$

$$\frac{P_1}{\gamma} + \frac{8^2}{2 \times 9.81} + 3 = \frac{1.39^2}{2 \times 9.81}$$

$\therefore$ $\frac{P_1}{\gamma}$ = **– 6.16 m of water.** (Below atmospheric pressure)

Additional head provided due to the use of draft tube will equal 6.16 m of water

**Note:** This may cause cavitation if the pressure is below the vapour pressure at the temperature condition. Though theoretically the pressure at turbine exit can be reduced to a low level, cavitation problem limits the design pressure.

## SOLVED PROBLEMS

**Problem 2.1** *A venturimeter is used to measure the volume flow. The pressure head is recorded by a manometer. When connected to a horizontal pipe the manometer reading was h cm. If the reading of the manometer is the same when it is connected to a vertical pipe with flow upwards and (ii) vertical pipe with flow downwards, discuss in which case the flow is highest.*

Consider equation 2.6.2

$$Q = \frac{A_2}{\left[1 - (A_2/A_1)^2\right]^{0.5}} \left[2gh\left(\frac{S_2}{S_1} - 1\right)\right]^{0.5}$$

As long as '$h$' remains the same, the volume flow is the same for a given venturimeter as this expression is a general one derived without taking any particular inclination.

This is because of the fact that the manometer automatically takes the inclination into account in indicating the value of $(Z_1 - Z_2)$.

**Problem 2.2** *Water flows at the rate of 600 l/s through a horizontal venturi with diameter 0.5 m and 0.245 m. The pressure gauge fitted at the entry to the venturi reads 2 bar.* ***Determine the throat pressure****. Barometric pressure is 1 bar.*

Using Bernoulli equation and neglecting losses

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2,\ P_1 = 2 \text{ bar (gauge)} = 3 \text{ bar (absolute) } 3 \times 10^5 \text{ N/m}^2$$

$$V_1 = \frac{Q}{(\pi \times d^2/4)} = \frac{0.6}{(\pi \times 0.5^2/4)} = 3.056 \text{ m/s} \quad \text{can also use}$$

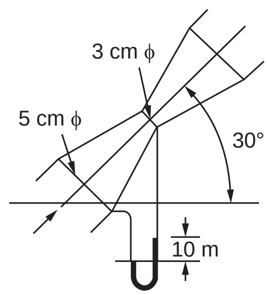
$$V_2 = V_1 \left(\frac{D_2}{D_1}\right)^2$$

$$V_2 = \frac{0.6}{(\pi \times 0.245^2/4)} = 12.732 \text{ m/s, Substituting}$$

$$\frac{3 \times 10^5}{9810} + \frac{3.056^2}{2 \times 9.81} + 0 = \frac{P_2}{9810} + \frac{12.732^2}{2 \times 9.81} + 0$$

$\therefore$ $\mathbf{P_2} = 223617 \text{ N/m}^2 =$ **2.236 bar (absolute) = 1.136 bar (gauge)**

**Problem 2.3** *A venturimeter as shown in Fig P. 6.3 is used measure flow of petrol with a specific gravity of 0.8. The manometer reads 10 cm of mercury of specific gravity 13.6.* ***Determine the flow rate.***




Using equation 2.6.2

$$Q = \frac{A_2}{\left[1 - (A_2/A_1)^2\right]^{0.5}} \left[2gh\left(\frac{S_2}{S_1} - 1\right)\right]^{0.5}$$

$A_2 = (\pi/4)\ 0.03^2$ as $D_2 = 3$ cm

$\therefore$ $(A_2/A_1)^2 = (D_2/D_1)^4 = (0.03/0.05)^4$,

$h = 0.10$ m $S_2 = 13.6$, $S_1 = 0.8$, Substituting,

$$\boldsymbol{Q} = \frac{(\pi \times 0.03^2/4)}{\left[1 - (0.03/0.05)^4\right]^{0.5}} \left[2 \times 9.81 \times 0.1\left(\frac{13.6}{0.8} - 1\right)\right]^{0.5}$$

= **$4.245 \times 10^{-3}$ m³/s or 15.282 m³/hr** or 4.245 l/s or 15282 l/hr or 3.396 kg/s

**Problem 2.4** *A liquid with specific gravity 0.8 flows at the rate of 3 l/s through a venturimeter of diameters 6 cm and 4 cm. If the manometer fluid is mercury (sp. gr = 13.b)* ***determine the value of manometer reading, h.***

Using equation (6.6.2)

$$Q = \frac{A_2}{\left[1-(A_2/A_1)^2\right]^{0.5}}\left[2gh\left(\frac{S_2}{S_1}-1\right)\right]^{0.5}$$

$$A_1 = \frac{\pi \times 0.06^2}{4} = 2.83 \times 10^{-3}\ \text{m}^2\ ;$$

$$A_2 = \frac{\pi \times 0.04^2}{4} = 1.26 \times 10^{-3}\ \text{m}^2$$

$$3 \times 10^{-3} = \frac{1.26 \times 10^{-3}}{\left[1-\left(\frac{1.26 \times 10^{-3}}{2.83 \times 10^{-3}}\right)^2\right]^{0.5}}\left[2 \times 9.81 \times h\left(\frac{13.6}{0.8}-1\right)\right]^{0.5}$$

Solving, **h = 0.0146 m = 14.6 mm.** of mercury column.

**Problem 2.5** *Water flows upwards in a vertical pipe line of gradually varying section from point 1 to point 2, which is 1.5m above point 1, at the rate of 0.9m³/s. At section 1 the pipe dia is 0.5m and pressure is 800 kPa. If pressure at section 2 is 600 kPa,* ***determine the pipe diameter at that location.*** *Neglect losses.*

Using Bernoulli equation,

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2$$

$$\frac{800 \times 10^3}{9810} + \frac{\left(0.9 \times 4/\pi \times 0.5^2\right)}{2 \times 9.81} + 0 = \frac{600 \times 10^3}{9810} + \frac{V_2^2}{2 \times 9.81} + 1.5$$

Solving, $V_2 = 19.37$ m/s.

$$\text{Flow = area × velocity, } \frac{\pi \times d_2^2}{4} \times 19.37 = 0.9\ \text{m}^3/\text{s}$$

Solving for $d_2$, **Diameter of pipe at section 2 = 0.243 m**

As ($p/\gamma$) is involved directly on both sides, gauge pressure or absolute pressure can be used without error. However it is desirable to use absolute pressure to aviod nagative pressure values.

**Problem 2.6** ***Calculate the exit diameter, if*** *at the inlet section of the draft tube the diameter is 1 m and the pressure is 0.405 bar absolute. The flow rate of water is 1600 l/s. The vertical distance between inlet and outlet is 6 m.*

Applying Bernoulli equation between points 1 and 2, neglecting losses

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2$$

$$V_1 = \frac{Q \times 4}{\pi \times D_1{}^2} = \frac{1600 \times 10^{-3} \times 4}{\pi \times 1^2} = 2.04 \text{ m/s}$$

$P_2$ = atmospheric pressure; $Z_2 = 0$ (datum); $Z_1 = 6$ m

$$\frac{0.405 \times 10^5}{9810} + \frac{2.04^2}{2 \times 9.81} + 6 = \frac{1.013 \times 10^5}{9810} + \frac{V_2^2}{2 \times 9.81} + 0 \quad \therefore V_2 = 0.531 \text{ m/s}$$

$$\frac{A_2}{A_1} = \frac{V_1}{V_2} = \frac{D_2^2}{1^2} = \frac{2.04}{0.531} \quad \therefore \quad \mathbf{D_2 = 1.96\ m}$$

0.405 bar absolute means vacuum at the inlet section of the draft tube. This may cause "cavitation" if this pressure is below the vapour pressure at that temperature. Though theoretically the pressure at turbine exit, where the draft tube is attached, can be reduced to a vary low level, cavitation problem limits the pressure level.

**Problem 2.7** *Water flows at the rate of 200 l/s upwards through a tapered vertical pipe. The diameter at the bottom is 240 mm and at the top 200 mm and the length is 5m. The pressure at the bottom is 8 bar, and the pressure at the topside is 7.3 bar.* ***Determine the head loss through the pipe****. Express it as a function of exit velocity head.*

Applying Bernoulli equation between points 1 (bottom) and 2 (top) and considering the bottom level as datum.

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2 + \text{losses}$$

$$\frac{8 \times 10^2}{9810} + \frac{(200 \times 10^{-3} \times 4)/\pi \times 0.24^2)^2}{2 \times 9.81} + 0$$

$$= \frac{7.3 \times 10^5}{9810} + \frac{(200 \times 10^{-3} \times 4)/(\pi \times 0.2^2)^2}{2 \times 9.81} + 5 + \text{losses}$$

$\therefore$ **Losses = 1.07 m**

$$1.07 = X\frac{V_2^2}{2g} = X\left[\frac{200 \times 10^{-3} \times 4}{\pi \times 0.22}\right]^2 /2 \times 9.81 \quad \therefore X = 0.516,$$

$$\textbf{Loss of head} = \mathbf{0.516\ \frac{V_2^2}{2g}}$$

**Problem 2.8** ***Calculate the flow rate of oil*** *(sp. gravity, 0.8) in the pipe line shown in Fig. P. 2.8. Also calculate* ***the reading "h"*** *shown by the differential manometer* ***fitted to the pipe*** *line which is filled with mercury of specific gravity 13.6.*

Applying Bernoulli equation (neglecting losses) between points 1 and 2

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2$$

$P_1 = 2 \times 10^5$ N/m$^2$; $P_2 = 0.8 \times 10^5$ N/m$^2$ ;

$Z_1 = 0$, $Z_2 = 2$ m

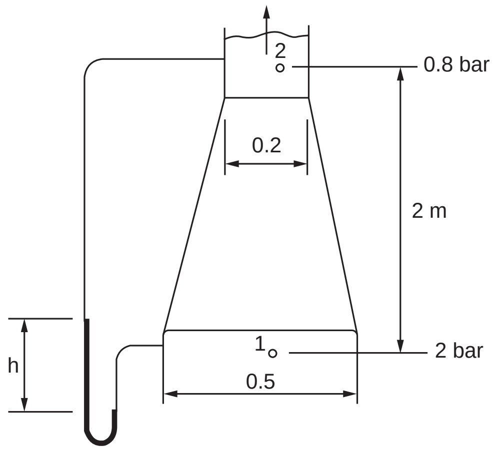



**Figure P. 2.8**

Applying continuity equation between points 1 and 2

$$A_1V_1 = A_2V_2,\ V_2 = V_1\frac{A_1}{A_2} = V_1\left(\frac{\pi \times 0.5^2/4}{\pi \times 0.2^2/4}\right) = 6.25\ V_1$$

$$\frac{2 \times 10^5}{9810 \times 0.8} + \frac{V_1^2}{2 \times 9.81} + 0 = \frac{0.8 \times 10^5}{9810 \times 0.8} + \frac{(6.25V_1)^2}{2 \times 9.81} + 2 \quad \therefore V_1 = 2.62 \text{ m/s}$$

Flow rate, $\mathbf{Q} = A_1\ V_1 = \dfrac{\pi \times 0.5^2}{4} \times 2.62 = 0.514$ m$^3$/s $=$ **514 l/s**

Using equation (6.6.2) (with $A_2 = 0.031$ m$^2$, $A_1 = 0.196$ m$^2$)

Flow rate,
$$Q = \frac{A_2}{\left[1-(A_2/A_1)^2\right]^{0.5}}\left[2gh\left(\frac{S_2}{S_1} - 1\right)\right]^{0.5}$$

$$0.514 = \frac{0.031}{\left[1-\left(\frac{0.031}{0.196}\right)^2\right]^{0.5}}\left[2 \times 9.81 \times h\left(\frac{13.6}{0.8} - 1\right)\right]^{0.5}$$

**Solving,** **h = 0.854 m**

**Problem 2.9** *Water flows at the rate of 400 l/s through the pipe with inlet (1) diameter of 35 cm and (2) outlet diameter of 30 cm with 4m level difference with point 1 above point 2. If $P_1$ = $P_2$ = 2 bar absolute,* ***determine the direction of flow.***

Consider datum as plane 2

$$\text{Total head 1,} \quad \frac{2\times10^5}{9810} + \frac{\left(0.4\times4/\pi\times0.35^2\right)^2}{2\times9.81} + 4 = 25.27 \text{ m water column}$$

$$\text{Total head at 2,} \quad \frac{2\times10^5}{9810} + \frac{\left(0.4\times4/\pi\times0.3^2\right)^2}{2\times9.81} + 0 = 22.02 \text{ m of water column}$$

The total energy at all points should be equal if there are no losses. This result shows that there are losses between 1 and 2 as the total energy at 2 is lower. **Hence the flow will take place from points 1 to 2**.

**Problem 2.10** *Petrol of relative density 0.82 flows in a pipe shown Fig. P.2.10. The pressure value at locations 1 and 2 are given as 138 kPa and 69 kPa respectively and point 2 is 1.2m vertically above point 1.* ***Determine the flow rate****. Also calculate* ***the reading of the differential manometer*** *connected as shown. Mercury with S = 13.6 is used as the manometer fluid.*

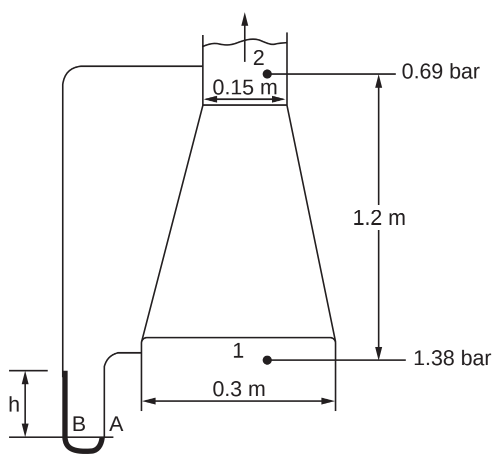



**Figure P. 2.10** Problem Model

Considering point 1 as a datum and using Bernoulli equation.

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2, Z_1 = 0, Z_2 = 1.2 \text{ m}, V_2 = V_1\frac{A_1}{A_2} = V_1\left(\frac{D_1^2}{D_2^2}\right)$$

$$\therefore \qquad V_2^{\,2} = V_1^{\,2}\left(\frac{D_1^4}{D_2^4}\right) = 16\, V_1^{\,2} \text{ as } D_1/D_2 = 2$$

$$\frac{138 \times 10^3}{0.82 \times 9810} + \frac{V_1^2}{2g} + 0 = \frac{69 \times 10^3}{0.82 \times 9810} + 16\left(\frac{V_1^2}{2g}\right) + 1.2$$

$$\frac{(138 - 69)10^3}{0.82 \times 9810} - 1.2 = 15\frac{V_1^2}{2g}. \quad \text{Solving, } V_1 = 3.106 \text{ m/s}$$

$\therefore$ **Volume flow** $= \frac{\pi \times 0.3^2}{4} \times 3.106 =$ **0.22 m³/s or 180 kg/s**

The flow rate is given by equation 6.6.2

$$Q = \frac{A_2}{\left[1 - \left(\frac{A_2}{A_1}\right)^2\right]^{0.5}}\left[2gh\left(\frac{S_2}{S_1} - 1\right)\right]^{0.5}, \quad \frac{S_2}{S_1} = \frac{13.6}{0.82}$$

$$0.22 = \frac{\pi \times 0.15^2/4}{\left[1 - \left(\frac{0.15}{0.3}\right)^4\right]^{0.5}}\left[2 \times 9.81 \times h\left(\frac{13.6}{0.82} - 1\right)\right]^{0.5}$$

**Solving,** **h = 0.475 m** of mercury column

**Problem 2.11** *Water flows downwards in a pipe as shown in Fig. P.6.11. If pressures at points 1 and 2 are to be equal,* ***determine the diameter of the pipe at point 2****. The velocity at point 1 is 6 m/s.*

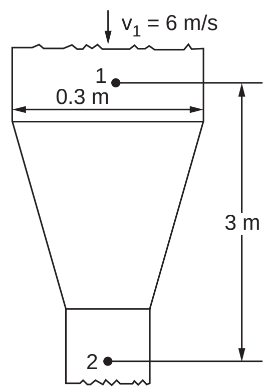



**Figure P. 2.11** Problem model

Applying Bernoulli equation between points 1 and 2 (taking level 2 as datum)

$$\frac{P_1}{\gamma} + \frac{6^2}{2 \times 9.81} + 3 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + 0$$

as $P_1 = P_2$, $\mathbf{V_2 = 9.74}$ **m/s**

Using the relation $A_1 V_1 = A_2 V_2$,

$$\frac{\pi \times 0.3^2 \times 6}{4} = \frac{\pi \times d^2 \times 9.74}{4}$$

$\therefore$ **d = 0.2355 m.**

**Problem 2.12** *A siphon is shown in Fig P. 2.12. Point A is 1m above the water level, indicated by point 1. The bottom of the siphon is 8m below level A. Assuming friction to be negligible,* ***determine the speed of the jet at outlet and also the pressure at A.***

Using Bernoulli equation, between 1 and 2.

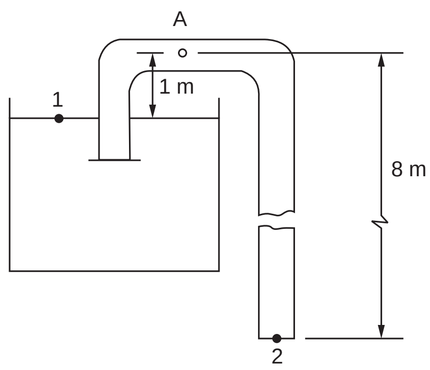



**Figure P. 2.12** Problem model

$$\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2 ,$$

$$P_1 = P_2 = \text{atmospheric pressure.}$$

Consider level 1 as datum. The velocity of water at the surface is zero.

$$\therefore \qquad 0 + 0 = \frac{V_2^2}{2g} - 7$$

$$\therefore \qquad \mathbf{V_2} = \sqrt{7 \times 2 \times 9.81} = \mathbf{11.72\ m/s = V_A}$$

Considering surface 1 and level $A$. As flow is the same,

$$\frac{P_1}{\gamma} + 0 + 0 = \frac{P_A}{\gamma} + 1 + \frac{V_A^2}{2g}$$

Considering $P_1/\gamma = 10.3$ m of water,

$$\frac{\mathbf{P_A}}{\gamma} = \frac{P_1}{\gamma} - 1 - \frac{V_2^2}{2g} = 10.3 - 1 - 7$$

= **2.3 m of water column** (absolute)

**Problem 2.13** *A pipe line is set up to draw water from a reservoir. The pipe line has to go over a barrier which is above the water level. The outlet is 8 m below water level.* ***Determine the maximum height of the barrier if the pressure*** *at this point should not fall below 1.0 m of water to avoid cavitation. Atmospheric pressure is 10.3 m.*

Considering outlet level 3 as datum and water level as 1 and appyling Bernoulli equation,

$$Z_3 = 0, Z_1 = 8, V_1 = 0, P_1 = P_3$$

$$\therefore \qquad 8 = \frac{V_3^2}{2g} \quad \therefore \quad V_3 = \sqrt{8 \times 9.81 \times 2} = 12.53 \text{ m/s}$$

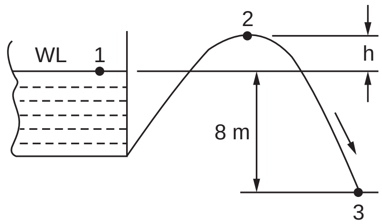



**Figure P. 2.13** Problem model

Considering the barrier top as level 2

$$\frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2 = \frac{P_3}{\gamma} + \frac{V_3^2}{2g} + Z_3, \quad \text{As } V_2 = V_3,\ Z_3 = 0,\ P_2/\gamma = 1$$

$$1 + Z_2 = 10.3$$

$\therefore$ $\mathbf{Z_2 = 9.3}$ **m.** Therefore the barrier can **be 1.3 m above water level.**

**Problem 2.14** ***Determine the flow rate of water*** *across the shutter in an open canal if the water level upstream of shutter is 5m and downstream is 2m. The width of the canal is 1m and flow is steady.*

Applying Bernoulli equation between point 1 in the upstream and point 2 in the downstream on both sides of the shutter, both surface pressures being atmospheric.

$$\frac{V_1^2}{2g} + 5 = \frac{V_2^2}{2g} + 2 \tag{1}$$

Applying continuity equation, flow rate, $Q = A_1V_1 = A_2V_2$

$(1 \times 5)\ V_1 = (1 \times 2)\ V_2$, $\because$ $V_2 = 2.5\ V_1$, Substituting in equation (1),

$$\frac{V_1^2}{2 \times 9.81} + 5 = \frac{(2.5V_1)^2}{2 \times 9.81} + 2,$$

$\therefore$ $V_1 = 3.35$ m/s, $V_2 = 8.37$ m/s. **Q = 16.742 m³/s,**

**Problem 2.15** *Uniform flow rate is maintained at a shutter in a wide channel. The water level in the channel upstream of shutter is 2m. Assuming uniform velocity at any section if the flow rate per m length is 3m³/s/m,* ***determine the level downstream****.*

Assume velocities $V_1$ and $V_2$ upstream and downstream of shutter and the datum as the bed level. Using Bernoulli equation

$$2 + \frac{V_1^2}{2g} = h_2 + \frac{V_2^2}{2g} \tag{A}$$

Considering unit width from continuity $1 \times 2 \times V_1 = 1 \times h_2 \times V_2$ (B)

$\therefore$ $V_2 = (2/h_2)\ V_1$, from flow rate $V_1 = 3/2 = 1.5$ m/s $\therefore$ $V_2 = \dfrac{3}{h_2}$

Substituting

$$2 + \frac{1.5^2}{2 \times 9.81} = h_2 + \frac{3^2}{h^2 \times 2 \times 9.81}$$

Simplifying, this reduces to $h_2^3 - 2.1147\, h_2^2 + 0.4587 = 0$

Solving, $h_2$ can be **2 m, – 0.425 m, 0.54 m**

**$h_2$ = 0.54 m is the acceptable answer.** 2m being trivial.

Using $B$, $0.54 \times V_2 \times 1 = 2 \times 1.5 = 3$. $\therefore V_2 = 5.56$ m/s.

check using $A$, $2 + 0.1147 = 0.54 + 1.57$ checks.

The difference between the dynamic head values will equal the difference between the datum heads. This may be checked using the calculated velocity values.

**Problem 2.16** *A pump with centre line 2m above the sump water level develops 50m head of water. The suction pipe is of 150 mm ID. The loss of head in the suction line is given by 5 $V_s^2/2g$. The delivery line is of 100 mm dia and the loss in the line is 12 $V_d^2/2g$. The water is delivered through a nozzle of 75 mm dia. The delivery is at 30m above the pump centre line.* ***Determine the velocity at the nozzle outlet and the pressure at the pump inlet*.**

Let the velocity at the nozzle be $V_n$

Velocity in the delivery pipe $= V_d = V_n \times \frac{75^2}{100^2} = \frac{9}{16} V_n$

Velocity in suction pipe $V_s = V_n \left(\frac{75}{150}\right)^2 = \frac{V_n}{4}$

Kinetic head at outlet $= \frac{V_n^2}{2g}$

Loss in delivery pipe $= \frac{V_d^2}{2g} = 12 \times \left(\frac{9}{16}\right)^2 \frac{V_n^2}{2g} = 3.797 \frac{V_n^2}{2g}$

Loss in suction pipe $= \frac{V_s^2}{2g} = \frac{5}{16} \frac{V_n^2}{2g} = 0.3125 \frac{V_n^2}{2g}$

Equating the head developed to the static head, losses and kinetic head,

$$50 = 30 + 2 + \frac{V_n^2}{2g} [1 + 3.797 + 0.3125]$$

$$18 \times 2 \times 9.81 = V_n^2 \, [5.109]$$

$\therefore$ **Velocity at the nozzle $V_n$ = 8.314 m/s**

**Pressure at suction** : Taking datum as the water surface and also the velocity of the water to be zero at the surface,

$P_1$ as atmospheric, 10.3 m of water column, Kinetic head $V^2/2g$, loss $5V^2/2g$

$$10.3 = \frac{P_2}{\gamma} + 2 + \left(\frac{(8.314/4)^2}{2 \times 9.81}\right) \times (5 + 1) \quad (\text{as } V_s = V_n/4)$$

$$\therefore \qquad \frac{\mathbf{P_2}}{\gamma} = 10.3 - 3.321 \text{ m} = \mathbf{6.979 \text{ m absolute}}$$

or 3.321 m below atmospheric pressure.

**Problem 2.17** *A liquid jet at a velocity $V_0$ is projected at angle θ. Describe the path of the free jet. Also calculate the* ***maximum height and the horizontal distance travelled****.*

The horizontal component of the velocity of jet is $V_{xo} = V_o \cos \theta$. The vertical component $V_{zo} = V_o \sin \theta$.

In the vertical direction, distance travelled, $Z$, during time $t$, (using the second law of Newton)

$$Z = V_{zo}\, t - (1/2)\, gt^2 \qquad \text{(A)}$$

The distance travelled along $x$ direction

$$X = V_{xo}\, t \text{ or } t = X/V_{xo} \qquad \text{(B)}$$

Solving for $t$ from $B$ and substituting in $A$,

$$Z = \frac{V_{zo}}{V_{xo}} X - \frac{1}{2} \frac{g}{V_{xo}^2} X^2 \qquad \text{(C)}$$

$Z$ value can be maximised by taking $dz/dx$ and equating to zero

$$\frac{dz}{dx} = \frac{V_{zo}}{V_{xo}} - \frac{1}{2} \frac{g}{V_{xo}^2} 2X, \quad \frac{V_{zo}}{V_{xo}} = \frac{gX}{V_{xo}^2} \quad \therefore \quad X = V_{zo}V_{xo}/g$$

Substituting in $C$,

$$Z_{max} = \frac{V_{zo}}{V_{xo}} \cdot \frac{V_{zo}V_{xo}}{g} - \frac{1}{2} \frac{g}{V_{xo}^2} \cdot \frac{V_{zo}^2 V_{xo}^2}{g^2}$$

$$= \frac{1}{2} \frac{V_{zo}^2}{g} = \frac{V_0{}^2 \sin^2 \theta}{2g}, \quad Z_{max} = V_o{}^2 \sin^2 \theta/2g \qquad \text{(D)}$$

The maximum height is achieved when θ = 90°.

$\therefore$ $\mathbf{X_{mas}}$ **= 2 times x as** $\mathbf{Z_{max.}}$

$$X_{max} = 2V_o{}^2 \sin \theta \cos \theta /g = V_o{}^2 \sin 2\theta/g \qquad \text{(E)}$$

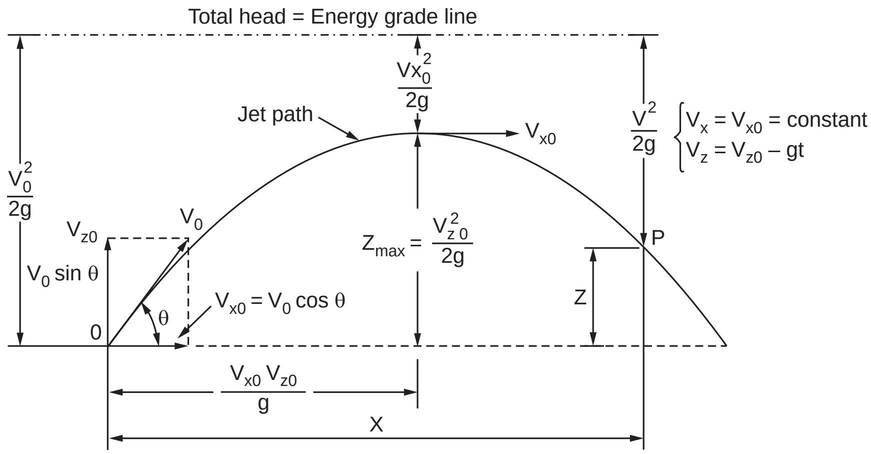
Maximum horizontal reach is at θ = 45° or 2θ = 90° and for this angle it will reach half the vertical height.

This describes an inverted parabola as shown in Fig. P.2.17

Bernoulli equation shows that $Z_t + V_t^2/2\text{g}$ = constant along the rejectory. $V_t$ is the velocity at that location when air drag is neglected. Pressure is assumed to be uniform all over the trejectory as it is exposed to atmosphere all along its travel. Hence

$Z_t + V_t^2 /2g$ = constant for the jet.

(**Note:** Velocity at time $t = V_{zo}\, t = V_0 \sin\theta + a \times t$, where $a = -g$, so the velocity decreases, becomes zero and then turns – ve)




**Problem 2.18** *A jet issuing at a velocity of 20 m/s is directed at 30° to the horizontal.* ***Calculate the height cleared*** *by the jet at 25m from the discharge location? Also* ***determine the maximum height the jet will clear*** *and the corresponding horizontal location.*

Ref Fig. P. 2.17

$$V_{xo} = V_o \cos 30 = 20 \cos 30 = 17.32 \text{ m/s};$$

$$V_{zo} = V_o \sin 30 = 20 \sin 30 = 10 \text{ m/s};$$

at time $t$, $X = V_{xo}t$; $Z = V_{zo}\, t - (1/2)\, gt^2$, Substituting for $t$ as $X/V_{xo}$ with $X = 25$ m

$$Z = \frac{V_{zo}}{V_{xo}} X - \frac{1}{2}\frac{g}{V_{xo}^2} X^2 \qquad \text{(A)}$$

Height cleared, $\quad \boldsymbol{Z_{25}} = \dfrac{10}{17.32} \times 25 - \dfrac{1}{2}\,\dfrac{9.81}{17.32^2} \times 25^2 = \mathbf{4.215\ m}$

**Maximum height of the jet trajectory** $= \dfrac{V_{zo}^2}{2g} = \dfrac{10^2}{2 \times 9.81} = \mathbf{5.097\ m}$

**Corresponding horizontal distance** $= \dfrac{V_{xo}V_{zo}}{g} = \dfrac{17.32 \times 10}{9.81} = \mathbf{17.66\ m}$

**Total horizontal distance is twice the distance travelled in reaching**

$\mathbf{Z_{max} = 35.32\ m}$

It would have crossed this height also at 10.43 m from the starting point (check using equations derived in Problem 6.17).

**Problem 2.19** ***Determine the velocity of a jet*** *directed at 40° to the horizontal to clear 6 m height at a distance of 20m. Also determine the maximum height this jet will clear and the total horizontal travel. What will be the horizontal distance at which the jet will be again at 6m height*.

From basics, referring to Fig. P. 2.17,

$$V_{xo} = V_o \cos 40, \quad V_{zo} = V_o \sin 40,$$

$$X = V_{xo}\, t,\ t = \frac{X}{V_{xo}}, \quad Z = V_{zo}\, t - (1/2)\, gt^2$$

Substituting for $t$ as $X/V_{xo}$

$$Z = \frac{V_{zo}}{V_{xo}} X - \frac{1}{2}\frac{g}{V_{xo}^2} X^2 \qquad (A)$$

Substituting the values,

$$6 = \frac{V_o \sin 40}{V_o \cos 40} \times 20 - \frac{1}{2} \times \frac{9.81 \times 20^2}{V_o^2 \cos^2 40}$$

$$6 = 20 \tan 40 - \frac{1}{2}\frac{9.81 \times 20^2}{V_o^2 \cos^2 40} \qquad (B)$$

$$\therefore \quad V_o^2 = \frac{9.81 \times 20^2}{2 \cos^2 40(20 \tan 40 - 6)} = 310 \quad \therefore \quad \mathbf{V_o = 17.61\ m/s.}$$

**Maximum height reached**

$$= V_{zo}^2/2g = (V_o \sin 40)^2/2g$$

$$= (17.61 \times \sin 40)^2 /2 \times 9.81 = \mathbf{6.53\ m}$$

The $X$ value corresponding to this is, (**half total horizontal travel**)

$$X = V_{xo}V_{zo}/g = 17.61^2 \sin 40 \cos 40/9.81 = \mathbf{15.56\ m.}$$

This shows that the jet clears 6m height at a distance of 20 m as it comes down. The jet would have cleared this height at a distance less than 15.56 m also. By symmetry, this can be calculated as – (20 – 15.56) + 15.56 = 11.12 m

check by substituting in equation $B$.

$$11.12 \tan 40 - \frac{1}{2} \times \frac{9.81 \times 11.12^2}{17.61^2 \cos^2 40} = 6$$

When both $Z$ and $X$ are specified unique solution is obtained. Given $V_o$ and $Z$, two values of $X$ is obtained from equation A.

**Problem 2.20** ***Determine the angle at which a*** *jet with a given velocity is to be projected for obtaining maximum horizontal reach*.

Refer Problem 6.17. $\quad X = V_{xo}\, t,\ Z = V_{zo}\, t - (1/2)\, gt^2$

The vertical velocity at any location/time is given by,

$$V_{zt} = \frac{dz}{dt} = V_{zo} - gt$$

The horizontal distance travelled will be half the total distance travelled when

$$V_{zt} = 0 \text{ or } t = V_{zo}/g$$

Total $X$ distance travelled during time $2t$.

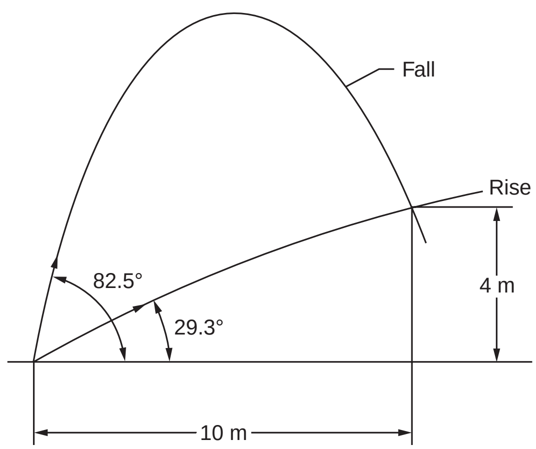
$$X = 2\, V_{xo}\, V_{zo}/g = 2\, V_o^2 \cos\theta \sin\theta/g = V_o^2 \sin 2\theta/g$$

For $X$ to be maximum sin 2θ should be maximum or 2θ = 90° or

**θ = 45°.** For maximum horizontal reach, the projected angle should be 45°.

The maximum reach, $X = V_o^2/g$ as sin 2θ = 1.

**Problem 2.21** ***Determined the angle at which*** *a jet with an initial velocity of 20 m/s is to be projected to clear 4m height at a distance of 10* m.




$$Z = \frac{V_{zo}}{V_{xo}} x - \frac{1}{2} \frac{gx^2}{V_{xo}^2}$$

Substituting in terms of $V_o$ and θ.

$$Z = \frac{V_o \sin\theta}{V_o \cos\theta} x - \frac{1}{2} \frac{gx^2}{V_o^2 \cos^2\theta} = x \tan\theta - \frac{1}{2} \frac{gx^2}{V_o^2} (\sec^2\theta)$$

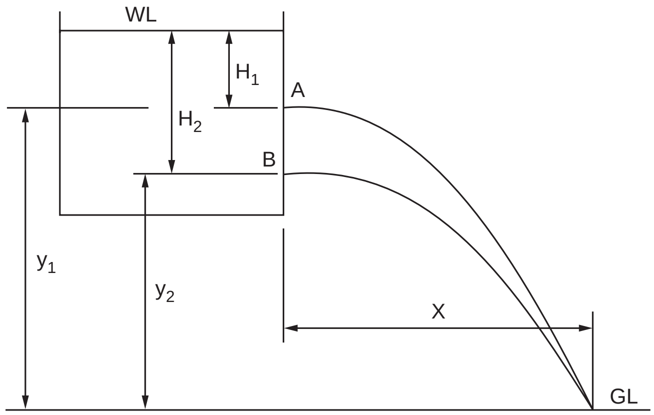
$$Z = x \tan\theta - \frac{1}{2} \frac{gx^2}{V_o^2} (1 + \tan^2\theta)$$

Substituting the given values, $4 = 10 \tan\theta - \dfrac{1}{2} \dfrac{9.81 \times 10^2}{20^2} (1 + \tan^2\theta)$

Hence, $\tan^2\theta - 8.155 \tan\theta + 4.262 = 0$, solving **tan θ = 7.594 or 0.5613**

**This corresponds to θ = 82.5° or 29.3°**. In the first case it clears the height during the fall. In the second case it clears the height during the rise. See Fig. P.6.21.

**Problem 2.22** *From a water tank two identical jets issue at distances $H_1$ and $H_2$ from the water level at the top. Both reach the same point at the ground level of the tank. If the* ***distance*** *from the ground level to the jet levels are $y_1$ and $y_2$.* ***Show that $H_1y_1 = H_2y_2$.***




In this case the jets issue out at *A* and *B* horizontally and so the position can be taken as the $Z_{max}$ position.

Referring to Problem 6.17, eqn. (*D*)

$$Z_{max} = \frac{V_{zo}^{\ 2}}{2g}, \quad y_1 = \frac{V_{zo1}^{\ 2}}{2g} \quad \text{or} \quad V_{zo1} = \sqrt{2gy_1}$$

Similarly, $$y_2 = \frac{V^2_{\ zo2}}{2g} \quad \text{or} \quad V_{zo2} = \sqrt{2gy_2} \qquad (A)$$

($V_{zo1}$ and $V_{zo2}$ are the *Z* components at point where the jet touches the ground)

$$X_{max} = \frac{V_{zo}V_{xo}}{g} \text{ and so } \frac{V_{zo1}V_{xo1}}{g} = \frac{V_{zo2}V_{xo2}}{g} \qquad (B)$$

$$V_{xo1} = \sqrt{2gH_1}\,,\ V_{xo2} = \sqrt{2gH_2} \qquad (C)$$

Substituting results (*A*) and (*C*) in equation (*B*), and simplifying,

$$\frac{\sqrt{2gH_1}\,\sqrt{2gy_1}}{g} = \frac{\sqrt{2gH_2}\,\sqrt{2gy_2}}{g} \qquad \therefore \quad \mathbf{H_1\,y_1 = H_2\,y_2}$$

**Problem 2.23** *A jet of water initially 12 cm dia when directed vertically upwards, reaches a maximum height of 20 m. Assuming the jet remains circular* ***determine the flow rate and area of jet at 10 m height****.*

As $V = 0$ at a height of 20 *m*, Bernoulli equation reduces to

$$\frac{V^2}{2g} = 20,$$

$\therefore \qquad V = (20 \times 9.81 \times 2)^{0.5} = 19.809$ m/s

$$\textbf{Flow rate} = \text{area} \times \text{velocity} = \frac{\pi \times 0.12^2}{4} \times 19.809 = \mathbf{0.224\ m^3/s}$$

When the jet reaches 10 m height, the loss in kinetic energy is equal to the increase in potential energy. Consider this as level 2 and the maximum height as level 1 and ground as datum,

$$P_1 = P_2,\ V_1 = 0,\ Z_2 = Z_1 - 10 = (20 - 10) = 10$$

$$20 = 10 + \frac{V^2{}_{20}}{2g} \quad \therefore \quad \frac{V_2{}^2}{2g} = 10,$$

$\therefore \qquad \mathbf{V_2} = (10 \times 2 \times 9.81)^{0.5} = \mathbf{14\ m/s}$

$$\textbf{Flow rate} = \text{area} \times \text{velocity},\ 0.224 = \frac{\pi \times D^2}{4} \times 14 \quad \therefore \quad \mathbf{D = 0.1427\ m}$$

**Problem 2.24** *Water is discharged through a 150 mm dia pipe fitted to the bottom of a tank. A pressure gauge fitted at the bottom of the pipe which is 10 m below the water level shows 0.5 bar. Determine the flow rate. Assume the frictional loss as* $4.5V_2{}^2/2g$.

Applying Bernoulli equation between the water level, 1 and the bottom of the pipe, 2 and this level as datum

$$\frac{P_1}{\gamma} + \frac{V_1{}^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2{}^2}{2g} + Z_2 + \text{losses}$$

$$0 + 0 + 10 = \frac{0.5 \times 10^5}{9810} + \frac{V_2{}^2}{2 \times 9.81} + 0 + 4.5\,\frac{V_2{}^2}{2 \times 9.81}$$

Solving, $\qquad \mathbf{V_2 = 4.18\ m/s}$

$$\textbf{Flow rate} = \frac{\pi \times 0.15^2}{4} \times 4.18 = \mathbf{0.0739\ \mathit{m^3/s} = 73.9\ l/s.}$$

**Problem 2.25** *An open tank of diameter D containing water to depth* $h_o$ *is emptied by a smooth orifice at the bottom. Derive an expression for the time taken to reduce the height to h. Also find the time* $t_{max}$ *for emptying the tank.*

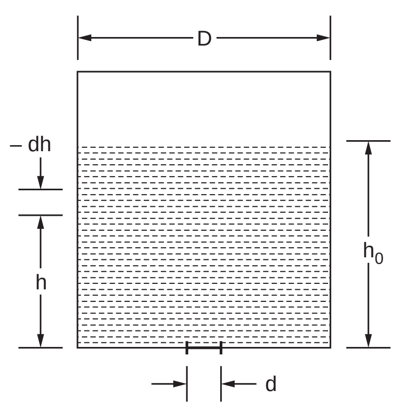



**Figure P. 2.25** Problem model

Considering point 1 at the top of the tank and point 2 at the orifice entrance, and point 2 as datum

$$P_{atm} + \frac{V_1{}^2}{2g} + h = \frac{V_2{}^2}{2g} + p_{atm}$$

$$\therefore \quad \frac{V_1{}^2}{2g} + h = \frac{V_2{}^2}{2g}$$

Also $\qquad V_1{}^2 = V_2{}^2\left[\frac{d}{D}\right]^4$

$$\therefore \qquad V_2 = \sqrt{\frac{2gh}{1-(d/D)^4}}$$

Let the level at the time considered be $h$.

The drop in level $dh$ during time $dt$ is given by (as $dh$ is negative with reference to datum)

$$\frac{dh}{dt} = -\frac{V_2 A_2}{A_1} = -\left(\frac{d}{D}\right)^2 \sqrt{\frac{2gh}{1-\left(\frac{d}{D}\right)^4}}$$

Taking $\left(\frac{d}{D}\right)^2$ inside and rearranging

$$\frac{dh}{dt} = -\sqrt{\frac{2gh}{\left(\frac{D}{d}\right)^4 - 1}}$$

Separating variables and integrating

$$\int_{h_o}^{h} \frac{dh}{\sqrt{h}} = -\sqrt{\frac{2g}{\left(\frac{D}{d}\right)^4 - 1}} \cdot \int_0^t dt$$

$$2\left[\sqrt{h_o} - \sqrt{h}\right] = \sqrt{\frac{2g}{\left(\frac{D}{d}\right)^4 - 1}} \, .\, t \qquad (A)$$

$$t = 2(\sqrt{h_o} - \sqrt{h}) / \sqrt{\frac{2g}{\left(\frac{D}{d}\right)^4 - 1}} = \sqrt{h_o} - \sqrt{h} / \sqrt{\frac{g/2}{\left(\frac{D}{d}\right)^4 - 1}} \qquad (B)$$

Equation ($A$) can be rearranged to give

$$\frac{h}{h_0} = \left[1 - \frac{t\sqrt{g/2\,h_o}}{\sqrt{\left(\frac{D}{d}\right)^4 - 1}}\right]^2 \qquad (C)$$

Equation ($B$) will be useful to find the drop in head during a given time interval.

Consider a numerical problem.

**Let** $\qquad$ **D = 0.5 m, d = 0.025 m, $h_o$ = 0.5 m**,

**Time for emptying is calculated as h = 0,**

$$t = \sqrt{h_o} / \sqrt{\frac{g/2}{\left(\frac{D}{d}\right)^4 - 1}}$$

$$= \sqrt{0.5}/\sqrt{\frac{9.81/2}{\left(\frac{0.5}{0.025}\right)^4 - 1}} = 127.7 \text{ seconds.}$$

To find the drop in level in say 100 seconds.

$$\frac{h}{h_o} = \left[1 - \frac{100\sqrt{9.81/2 \times 0.5}}{\sqrt{\left(\frac{0.5}{0.25}\right)^4 - 1}}\right]^2 = 0.0471$$

$\therefore$ Drop in head = 0.5 (1 – 0.0471) = 0.4764 m

In case $d << D$, then $V_2 = \sqrt{2gh}$ when head is $h$ m

$$\frac{dh}{dt} = -\frac{A_2V_2}{A_1} = -V_2\left(\frac{d}{D}\right)^2 = -\left(\frac{d}{D}\right)^2 . \sqrt{2gh}$$

Separating variables and integrating

$$\int_{h_o}^{h} \frac{dh}{\sqrt{h}} = -\left(\frac{d}{D}\right)^2 \sqrt{2g} . \int_0^t dt$$

$$2\,[\sqrt{h_o} - \sqrt{h}\,] = \left(\frac{d}{D}\right)^2 \sqrt{2g} . t$$

In this case to empty the tank,

$$2\sqrt{0.5} = \left(\frac{0.025}{0.5}\right)^2 . \sqrt{2 \times 9.81} . t.$$

**Solving** **t = 127.71 s**.

The same answer because the same diameter of the orifice is used. Say $d$ = 0.01 m, then time for employing is 1130 sec.

**Problem 2.26** *Two identical jets issuing from a touch as shown in figure reach the ground at a distance of 10 m.* ***Determine the distances indicated as h and H***.

Consider top jet:

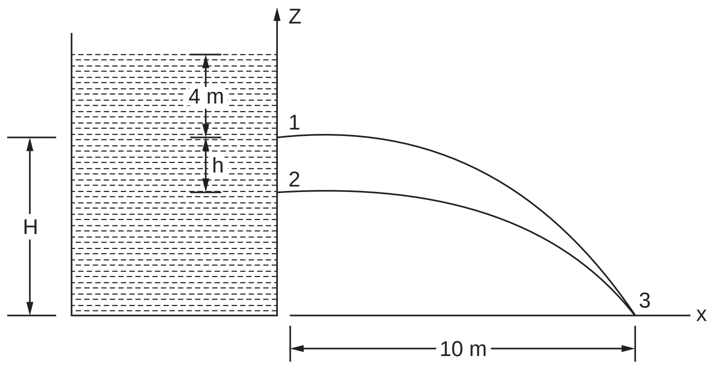
$x$ distance travelled in time $t$ is 10 m.

$$\therefore \quad V_{xo1}t = 10 \qquad (A)$$

$$t = 10/V_{xo1}$$

The height drop is as $V_{zo}$ as start is zero,

$$\therefore \quad V_{zo1}t = H = \frac{1}{2}gt^2 \qquad (B)$$




$$H = \frac{1}{2} g \frac{100}{V_{xo2}} \quad \therefore \quad V^2_{xo} = \frac{50g}{H}$$

As jet issues from the nozzle it has any $x$ directional velocity $V_{xo1}$, is present.

$$V^2_{xo1} = 2\,g4 = 8g \qquad (C) \text{ (as head available in 4 m)}$$

Substituting, $\quad 8g = \frac{50g}{H} \quad$ or $\quad$ **H = 6.25 m.**

Considering the second jet.

$$V_{xo2}\, t = 10,\ t = \frac{10}{V_{xo2}},$$

The head drop in $(H - h)$ m. As in the previous case $V_{zoc} = 0$ at start

$$H - h = \frac{1}{2} gt^2. \text{ Substituting}$$

$$H - h = \frac{1}{2} g \frac{1V_o}{V_{xo2}{}^2} = \frac{50g}{V_{xo2}{}^2} \qquad (D)$$

As at start only $V_{xo2}$ is present,

$$V_{xo2}{}^2 = (4 + h)\, g \times 2$$

Substituting in $(D)$

$$H - h = \frac{50g}{(4+h)\, g \times 2} = \frac{25}{4+h}, \text{ as } \mathbf{H = 6.25\ m.}$$

$$6.25 - h = \frac{25}{4+h}. \text{ This leads to}$$

$h^2 - 2.25h = 0$, $\quad$ or $\quad$ **h = 2.25 m.**

It may be also noted from problem 6.22.

$$H \times 4 = (H - h)\,(4 + h).$$

$$6.25 \times 4 = 4 \times 6.25$$

Hence this condition is also satisfied.