

## Boundary conditions for Electrostatic fields

In our discussions so far we have considered the existence of electric field in the homogeneous medium. Practical electromagnetic problems often involve media with different physical properties. Determination of electric field for such problems requires the knowledge of the relations of field quantities at an interface between two media. The conditions that the fields must satisfy at the interface of two different media are referred to as *boundary conditions* .

In order to discuss the boundary conditions, we first consider the field behavior in some common material media.

In general, based on the electric properties, materials can be classified into three categories: conductors, semiconductors and insulators (dielectrics). In *conductor* , electrons in the outermost shells of the atoms are very loosely held and they migrate easily from one atom to the other. Most metals belong to this group. The electrons in the atoms of *insulators* or *dielectrics* remain confined to their orbits and under normal circumstances they are not liberated under the influence of an externally applied field. The electrical properties of *semiconductors* fall between those of conductors and insulators since semiconductors have very few numbers of free charges.

The parameter *conductivity* is used characterizes the macroscopic electrical property of a material medium. The notion of conductivity is more important in dealing with the current flow and hence the same will be considered in detail later on.

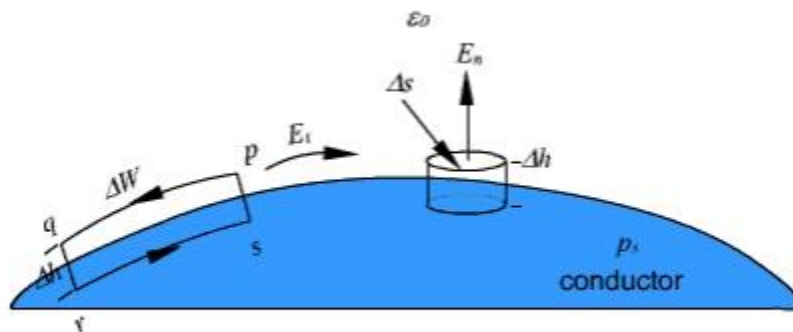
If some free charge is introduced inside a conductor, the charges will experience a force due to mutual repulsion and owing to the fact that they are free to move, the charges will appear on the surface. The charges will redistribute themselves in such a manner that the field within the conductor is zero. Therefore, under steady condition, inside a conductor

$$\rho_v = 0$$

From Gauss's theorem it follows that

$$\vec{E} = 0 \dots\dots\dots(2.51)$$

The surface charge distribution on a conductor depends on the shape of the conductor. The charges on the surface of the conductor will not be in equilibrium if there is a tangential component of the electric field is present, which would produce movement of the charges. Hence under static field conditions, tangential component of the electric field on the conductor surface is zero. **The electric field on the surface of the conductor is normal everywhere to the surface** . Since the tangential component of electric field is zero, the **conductor surface is an equipotential surface**. As



$$\vec{E} = 0$$

inside the conductor, the conductor as a whole has the same potential. We may further note that charges require a finite time to redistribute in a conductor. However, this time is very small  $\sim 10$ - $15$  sec for good conductor like copper.

**Fig 2.14: Boundary Conditions for at the surface of a Conductor**

Let us now consider an interface between a conductor and free space as shown in the figure 2.14. Let us consider the closed path  $pqrsp$  for which we can write,

$$\oint \vec{E} \cdot d\vec{l} = 0 \dots\dots\dots(2.52)$$

For  $\Delta h \rightarrow 0$  and noting that  $\vec{E}$  inside the conductor is zero, we can write

$$E_t \Delta w = 0 \dots\dots\dots(2.53)$$

$E_t$  is the tangential component of the field. Therefore we find that

$$E_t = 0 \dots\dots\dots(2.54)$$

In order to determine the normal component  $E_n$ , the normal component of  $\vec{E}$ , at the surface of the conductor, we consider a small cylindrical Gaussian surface as shown in the Fig.12. Let  $\Delta s$  represent the area of the top and bottom faces and  $\Delta h$  represents the height of the cylinder. Once again, as  $\Delta h \rightarrow 0$ , we approach the surface of the conductor. Since  $\vec{E} = 0$  inside the conductor is zero,

$$\epsilon_0 \oint \vec{E} \cdot d\vec{s} = \epsilon_0 E_n \Delta s = \rho_s \Delta s \dots\dots\dots(2.55)$$

$$E_n = \frac{\rho_s}{\epsilon_0} \dots\dots\dots(2.56)$$

Therefore, we can summarize the boundary conditions at the surface of a conductor as:

$$E_t = 0 \dots\dots\dots(2.57)$$

$$E_n = \frac{\rho_s}{\epsilon_0} \dots\dots\dots(2.58)$$

**Behavior of dielectrics in static electric field: Polarization of dielectric**

Here we briefly describe the behavior of dielectrics or insulators when placed in static electric field. Ideal dielectrics do not contain free charges. As we know, all material media are composed of atoms where a positively charged nucleus (diameter  $\sim 10^{-15}\text{m}$ ) is surrounded by negatively charged electrons (electron cloud has radius  $\sim 10^{-10}\text{m}$ ) moving around the nucleus. Molecules of dielectrics are neutral macroscopically; an externally applied field causes small displacement of the charge particles creating small electric dipoles. These induced dipole moments modify electric fields both inside and outside dielectric material.

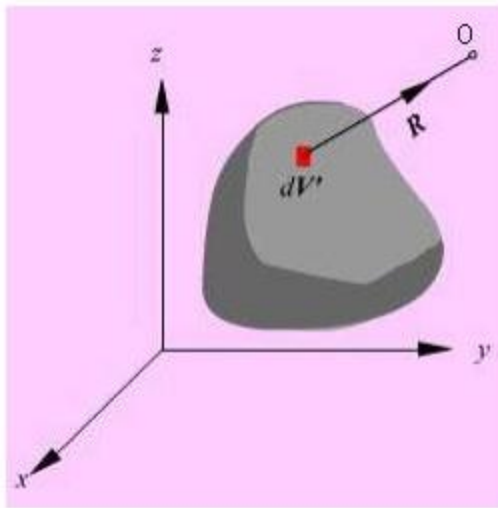
Molecules of some dielectric materials possess permanent dipole moments even in the absence of an external applied field. Usually such molecules consist of two or more dissimilar atoms and are called *polar* molecules. A common example of such molecule is water molecule  $H_2O$ . In polar molecules the atoms do not arrange themselves to make the net dipole moment zero. However, in the absence of an external field, the molecules arrange themselves in a random manner so that net dipole moment over a volume becomes zero. Under the influence of an applied electric field, these dipoles tend to align themselves along the field as shown in figure 2.15. There are some materials that can exhibit net permanent dipole moment even in the absence of applied field. These materials are called *electrets* that made by heating certain waxes or plastics in the presence of electric field. The applied field aligns the polarized molecules when the material is in the heated state and they are frozen to their new position when after the temperature is brought down to its normal temperatures. Permanent polarization remains without an externally applied field.



As a measure of intensity of polarization, polarization vector  $\vec{P}$  (in C/m<sup>2</sup>) is

defined as: 
$$\vec{P} = \lim_{\Delta v \rightarrow 0} \frac{\sum_{k=1}^{n \Delta v} \vec{P}_k}{\Delta v} \dots\dots\dots(2.59)$$

$n$  being the number of molecules per unit volume i.e.  $\vec{P}$  is the dipole moment per unit volume. Let us now consider a dielectric material having polarization  $\vec{P}$  and compute the potential at an external point O due to an elementary dipole  $\vec{P} dv'$ .



**Fig 2.16: Potential at an External Point due to an Elementary Dipole  $\vec{P} dv'$ .**



With reference to the figure 2.16, we can write:  

$$dV = \frac{\vec{P} dv' \hat{a}_R}{4\pi\epsilon_0 R^2}$$
 .....(2.60)

Therefore,

$$V = \int_V \frac{\vec{P} \cdot \hat{a}_R}{4\pi\epsilon_0 R^2} dv' \dots\dots\dots(2.61)$$

$$R = \left\{ (x-x')^2 + (y-y')^2 + (z-z')^2 \right\}^{1/2} \dots\dots\dots(2.62)$$

where x,y,z represent the coordinates of the external point O and x',y',z' are the coordinates of the source point.

From the expression of R, we can verify that

$$\nabla' \left( \frac{1}{R} \right) = \frac{\hat{a}_R}{R^2} \dots\dots\dots(2.63)$$

$$V = \frac{1}{4\pi\epsilon_0} \int_V \vec{P} \cdot \nabla' \left( \frac{1}{R} \right) dv' \dots\dots\dots(2.64)$$

Using the vector identity,  $\nabla' \cdot (f\vec{A}) = f\nabla' \cdot \vec{A} + \vec{A} \cdot \nabla' f$ , where f is a scalar quantity, we have,

$$V = \frac{1}{4\pi\epsilon_0} \left[ \int_V \nabla' \cdot \left( \frac{\vec{P}}{R} \right) dv' - \int_V \frac{\nabla' \cdot \vec{P}}{R} dv' \right] \dots\dots\dots(2.65)$$

Converting the first volume integral of the above expression to surface integral, we can write

$$V = \frac{1}{4\pi\epsilon_0} \oint_S \frac{\vec{P} \cdot \hat{a}'_n}{R} ds' + \frac{1}{4\pi\epsilon_0} \int_V \frac{(-\nabla' \cdot \vec{P})}{R} dv' \dots\dots\dots(2.66)$$

where  $\hat{a}'_n$

is the outward normal from the surface element  $ds'$  of the dielectric. From the above expression we find that the electric potential of a polarized dielectric may be found from the contribution of volume and surface charge distributions having densities

$$\rho_{ps} = \vec{P} \cdot \hat{a}_n \dots\dots\dots(2.67)$$

$$\rho_{pv} = -\nabla \cdot \vec{P} \dots\dots\dots(2.68)$$

These are referred to as polarisation or bound charge densities. Therefore we may replace a polarized dielectric by an equivalent polarization surface charge density and a polarization volume charge density. We recall that bound charges are those charges that are not free to move within the dielectric material, such charges are result of displacement that occurs on a molecular scale during polarization. The total bound charge on the surface is

$$\oint_S \rho_{ps} ds = \oint_S \vec{P} \cdot d\vec{s} \dots\dots\dots(2.69)$$

The charge that remains inside the surface is

$$\int_V \rho_{pv} dv = \int_V -\nabla \cdot \vec{P} dv \dots\dots\dots(2.70)$$

The total charge in the dielectric material is zero as

$$\oint_S \rho_{ps} ds + \int_V \rho_{pv} = \oint_S \vec{P} \cdot d\vec{s} + \int_V -\nabla \cdot \vec{P} dv = \int_V \nabla \cdot \vec{P} - \int_V \nabla \cdot \vec{P} = 0 \dots\dots\dots(2.71)$$



If we now consider that the dielectric region containing charge density  $\rho_v$  the total volume charge density becomes

$$\rho_t = \rho_v + \rho_{pv} \dots\dots\dots(2.72)$$

Since we have taken into account the effect of the bound charge density, we can write

$$\nabla \cdot \vec{E} = \frac{(\rho_v + \rho_{pv})}{\epsilon_0} \dots\dots\dots(2.73)$$

Using the definition of  $\rho_{pv}$  we have

$$\nabla \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_v \dots\dots\dots(2.74)$$

Therefore the electric flux density  $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

When the dielectric properties of the medium are linear and isotropic, polarisation is directly proportional to the applied field strength and

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \dots\dots\dots(2.75)$$

is the electric susceptibility of the dielectric. Therefore,

$$\vec{D} = \epsilon_0 (1 + \chi_e) \vec{E} = \epsilon_0 \epsilon_r \vec{E} = \epsilon \vec{E} \dots\dots\dots(2.76)$$

$\epsilon_r = 1 + \chi_e$  is called relative permeability or the dielectric constant of the medium.  $\epsilon_0 \epsilon_r$  is called the absolute permittivity.

A dielectric medium is said to be linear when  $\chi_e$  is independent of  $\vec{E}$  and the medium is homogeneous if  $\chi_e$  is also independent of space coordinates. A



linear homogeneous and isotropic medium is called a **simple medium** and for such medium the relative permittivity is a constant.

Dielectric constant  $\epsilon_r$  may be a function of space coordinates. For anisotropic materials, the dielectric constant is different in different directions of the electric field, D and E are related by a permittivity tensor which may be written as:

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \dots\dots\dots(2.77)$$

For crystals, the reference coordinates can be chosen along the principal axes, which make off diagonal elements of the permittivity matrix zero. Therefore, we have

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \dots\dots\dots(2.78)$$

Media exhibiting such characteristics are called **biaxial**. Further, if  $\epsilon_1 = \epsilon_2$  then the medium is called **uniaxial**. It may be noted that for isotropic media,  $\epsilon_1 = \epsilon_2 = \epsilon_3$ .

Lossy dielectric materials are represented by a complex dielectric constant, the imaginary part of which provides the power loss in the medium and this is in general dependant on frequency.

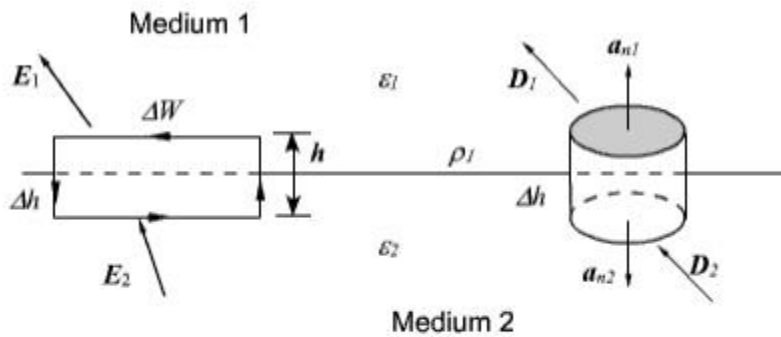
Another phenomenon is of importance is **dielectric breakdown**. We observed that the applied electric field causes small displacement of bound charges in a dielectric material that results into polarization. Strong field can pull electrons completely out of the molecules. These electrons being accelerated under influence of electric field will collide with molecular lattice structure causing damage or distortion of material. For very strong fields, avalanche breakdown may also occur. The dielectric under such condition will become conducting.

The maximum electric field intensity a dielectric can withstand without breakdown is referred to as the **dielectric strength** of the material.

**Boundary Conditions for Electrostatic Fields:**

Let us consider the relationship among the field components that exist at the interface between two dielectrics as shown in the figure 2.17. The permittivity of the medium 1 and medium 2 are

$\epsilon_1$  and  $\epsilon_2$  respectively and the interface may also have a net charge density  $\rho_s$  Coulomb/m.



**Fig 2.17: Boundary Conditions at the interface between two dielectrics**

We can express the electric field in terms of the tangential and normal

components 
$$\vec{E}_1 = \vec{E}_{1t} + \vec{E}_{1n}$$

$$\vec{E}_2 = \vec{E}_{2t} + \vec{E}_{2n} \dots\dots\dots(2.79)$$

where  $E_t$  and  $E_n$  are the tangential and normal components of the electric field respectively.

Let us assume that the closed path is very small so that over the elemental path length the variation of E can be neglected. Moreover very near to the interface,

$\Delta h \rightarrow 0$  . Therefore

$$\oint \vec{E} \cdot d\vec{l} = E_{1t} \Delta w - E_{2t} \Delta w + \frac{h}{2} (E_{1n} + E_{2n}) - \frac{h}{2} (E_{1n} + E_{2n}) = 0 \dots\dots\dots(2.80)$$

Thus, we have,

$$E_{1t} = E_{2t} \text{ or } \frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$$

i.e. the **tangential component of an electric field is continuous across the interface.**

For relating the flux density vectors on two sides of the interface we apply Gauss's law to a small pillbox volume as shown in the figure. Once again as  $\Delta h > 0$ , we can write

$$\oint \vec{D} \cdot d\vec{s} = (\vec{D}_1 \cdot \hat{a}_{n2} + \vec{D}_2 \cdot \hat{a}_{n1}) \Delta s = \rho_s \Delta s \dots\dots\dots(2.81a)$$

$$\text{i.e., } D_{1n} - D_{2n} = \rho_s \dots\dots\dots(2.81b)$$

$$\text{.e., } \epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s \dots\dots\dots(2.81c)$$

Thus we find that the **normal component of the flux density vector  $D$  is discontinuous across an interface by an amount of discontinuity equal to the surface charge density at the interface.**

**Example**

Two further illustrate these points; let us consider an example, which involves the refraction of  $D$  or  $E$  at a charge free dielectric interface as shown in the figure 2.18.

Using the relationships we have just derived, we can write

$$E_{1t} = E_1 \sin \theta_1 = \frac{D_1}{\epsilon_1} \sin \theta_1 = E_{2t} = E_2 \sin \theta_2 = \frac{D_2}{\epsilon_2} \sin \theta_2 \dots\dots\dots(2.82a)$$

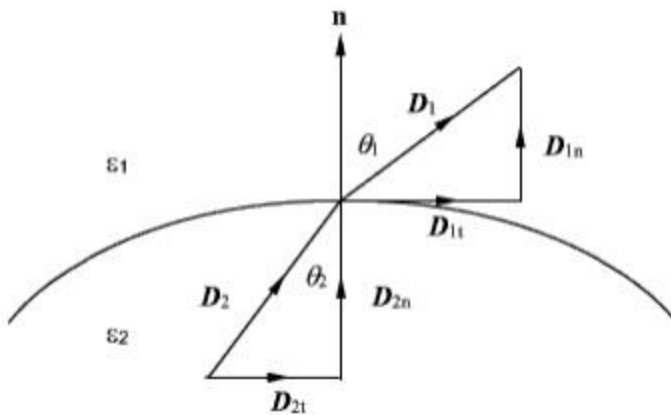
$$D_{1n} = D_1 \cos \theta_1 = D_{2n} = D_2 \cos \theta_2 \dots\dots\dots(2.82b)$$

In terms of flux density vectors,

$$\frac{D_1}{\epsilon_1} \sin \theta_1 = \frac{D_2}{\epsilon_2} \sin \theta_2 \dots\dots\dots(2.83a)$$

$$D_1 \cos \theta_1 = D_2 \cos \theta_2 \dots\dots\dots(2.83b)$$

Therefore,  $\frac{\tan \theta_1}{\tan \theta_2} = \frac{\epsilon_1}{\epsilon_2} = \frac{\epsilon_{r1}}{\epsilon_{r2}} \dots\dots\dots(2.84)$



**Fig 2.18: Refraction of D or E at a Charge Free Dielectric Interface**

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