Unit-2

ANALYSIS OF LTI SYSTEMS

Fourier series:

To represent any periodic signal x(t), Fourier developed an expression called Fourier series. This is in terms of an infinite sum of sines and cosines or exponentials. Fourier series uses orthogonality condition.

Jean Baptiste Joseph Fourier, a French mathematician and a physicist; was born in Auxerre, France. He initialized Fourier series; Fourier transforms and their applications to problems of heat transfer and vibrations. The Fourier series, Fourier transforms and Fourier's Law are named in his honor.

Fourier Series Representation of Continuous Time Periodic Signals

A signal is said to be periodic if it satisfies the condition

x (t) = x (t + T) or x (n) = x (n + N).

Where T = fundamental time period,

ω_0= fundamental frequency = 2π/T

There are two periodic signals

$x(t)=\cos\omega_0 t$(sinusoidal)

&$x(t)=e^{j\omega_0 t}$(complex exponential)

These two signals are periodic with period $T=2\pi/\omega_0$

. A set of harmonically related complex exponentials can be represented as $\{\varphi_k(t)\}$

$$\phi_k(t) = \{e^{jk\omega_0 t}\} = \{e^{jk(\frac{2\pi}{T})t}\} \text{where}\, k = 0 \pm 1, \pm 2 .. n \; ..... (1)$$

All these signals are periodic with period T

According to orthogonal signal space approximation of a function x (t) with n, mutually orthogonal functions is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \ldots\ldots(2)$$

$$= \sum_{k=-\infty}^{\infty} a_k k e^{jk\omega_0 t}$$

Where a_k= Fourier coefficient = coefficient of approximation. This signal x(t) is also periodic with period T.

Equation 2 represents Fourier series representation of periodic signal x(t).

The term k = 0 is constant.

The term $k=\pm1$ having fundamental frequency ω_0 , is called as 1st harmonics.

The term $k=\pm2$ having fundamental frequency $2\omega_0$, is called as 2nd harmonics, and so on... The term $k=\pm n$ having fundamental frequency $n\omega_0$, is called as nth harmonics.

Deriving Fourier Coefficient

We know that,

$$x(t) = \Sigma_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \ldots\ldots(1)$$

Multiplying e^{-jwot} on both sides

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.e^{-jn\omega_0 t}$$

Consider integral on both sides.

$$\int_0^T x(t)e^{jk\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.e^{-jn\omega_0 t}dt$$

$$= \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}.dt$$

$$\int_0^T x(t)e^{jk\omega_0 t}dt = \sum_{k=-\infty}^{\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}dt. \;\ldots\ldots(2)$$

by Euler's formula,

$$\int_0^T e^{j(k-n)\omega_0 t}dt. = \int_0^T \cos(k-n)\omega_0 dt + j\int_0^T \sin(k-n)\omega_0 t\,dt$$

$$\int_0^T e^{j(k-n)\omega_0 t}dt. = \begin{cases} T & k = n \\ 0 & k \neq n \end{cases}$$

Hence in equation 2, the integral is zero for all values of k except at k = n. Put k = n in equation 2.

2.

$$\Rightarrow \int_0^T x(t)e^{-jn\omega_0 t}\,dt = a_n T$$

$$\Rightarrow a_n = \frac{1}{T}\int_0^T e^{-jn\omega_0 t}\,dt$$

Replace n by k,

$$\Rightarrow a_k = \frac{1}{T}\int_0^T e^{-jk\omega_0 t}\,dt$$

$$\therefore x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}$$

$$\text{where} a_k = \frac{1}{T}\int_0^T e^{-jk\omega_0 t}\,dt$$

Fourier Transform

The main drawback of Fourier series is, it is only applicable to periodic signals. There are some naturally produced signals such as nonperiodic or aperiodic, which we cannot represent using Fourier series. To overcome this shortcoming, Fourier developed a mathematical model to transform signals between time (or spatial) domain to frequency domain & vice versa, which is called 'Fourier transform'.

Fourier transform has many applications in physics and engineering such as analysis of LTI systems, RADAR, astronomy, signal processing etc.

Deriving Fourier transform from Fourier series:

Consider a periodic signal f(t) with period T. The complex Fourier series representation of f(t) is given as,

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi}{T_0}kt} \ldots\ldots(1)$$

Let $\frac{1}{T_0} = \Delta f$, then equation 1 becomes

$$f(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k\Delta f t} \ldots\ldots(2)$$

but you know that

$$a_k = \frac{1}{T_0}\int_{t_0}^{t_0+T} f(t)e^{-jk\omega_0 t}\,dt$$

Substitute in equation 2.

$$2 \Rightarrow f(t) = \Sigma_{k=-\infty}^{\infty} \frac{1}{T_0}\int_{t_0}^{t_0+T} f(t)e^{-jk\omega_0 t}\,dt\, e^{j2\pi k\Delta f t}$$

Let $t_0 = \frac{T}{2}$

$$= \Sigma_{k=-\infty}^{\infty} \left[\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)e^{-j2\pi k\Delta f t}\,dt\right] e^{j2\pi k\Delta f t} . \Delta f$$

In the limit as $T\rightarrow\infty$, Δf approaches differential df, $k\Delta f$ becomes a continuous variable f, and summation becomes integration

$$f(t) = lim_{T\to\infty} \left\{ \Sigma_{k=-\infty}^{\infty} [\int_{\frac{-T}{2}}^{\frac{T}{2}} f(t)e^{-j2\pi k\Delta ft}\, dt]\, e^{j2\pi k\Delta ft} . \Delta f \right\}$$

$$= \int_{-\infty}^{\infty} [\int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}\, dt] e^{j2\pi ft}\, df$$

$$f(t) = \int_{-\infty}^{\infty} F[\omega]e^{j\omega t}\, d\omega$$

$$\text{Where } F[\omega] = [\int_{-\infty}^{\infty} f(t)e^{-j2\pi ft}\, dt]$$

Fourier transform of signal,

$$f(t) = F[\omega] = [\int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt]$$

Inverse Fourier transform of the signal

$$f(t) = \int_{-\infty}^{\infty} F[\omega]e^{j\omega t}\, d\omega$$

Conditions for Existence of Fourier Transform:

Any function f(t) can be represented by using Fourier transform only when the function satisfies Dirichlet's conditions. i.e.

- The function f(t) has finite number of maxima and minima.
- There must be finite number of discontinuities in the signal f(t), in the given interval of time.

It must be absolutely integrable in the given interval of time

$$\int_{-\infty}^{\infty} |\, f(t)|\, dt < \infty$$