ROHINI COLLEGE OF ENGINEERING & TECHNOLOGY

DEPARTMENT OF MATHEMATICS

UNIT - II : FOURIER SERIES

2.2 GENERAL FORM OF FOURIER SERIES

PART A

1. **State Dirichlet's conditions for the existence of Fourier series of *f(x)* in the interval $(0,2\pi)$.**

A function f(x) can be expanded as a Fourier series in the interval $(0,2\pi)$ if the following conditions are satisfied.

(i) f(x) is periodic, single valued and finite in $(0,2\pi)$

(ii) f(x) has only finite number of finite discontinuities and no infinite discontinuities in $(0,2\pi)$.

(iii) f(x) has only finite number of maxima and minima in $(0,2\pi)$

2. **Does $f(x) = \tan x$ possess a Fourier expansion?**

Solution:

$\tan x$ does not possess a Fourier expansion because the function $f(x) = \tan x = \dfrac{\sin x}{\cos x}$ has the infinite discontinuity at the point $x = \dfrac{\pi}{2}$.

3. **Determine the value of a_n & a_0 in the Fourier series expansion of $f(x) = x^3$ in $-\pi < x < \pi$**

Solution:

$f(x) = x^3 \Rightarrow f(-x) = (-x)^3 = -x^3 = -f(x) \Rightarrow f(x)$ is an odd function $\therefore\ a_n = a_0 = 0$

4. **Find the Fourier constant b_n for *x sin x* in $-\pi < x < \pi$, when expressed as a Fourier series.**

Solution:

$f(x) = x\sin x,\ -\pi < x < \pi$

$f(-x) = (-x)\sin(-x) = x\sin x = f(x)$

$\therefore\ f(x)$ is an even function $\therefore b_n = 0$

5. **Find the constant term of the Fourier series for the function $f(x) = x^2$, $-\pi < x < \pi$**

Solution:

$f(x) = x^2, -\pi < x < \pi$

$f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore$ *f(x)* is an even function

$$l = \frac{U.L - L.L}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} = \pi$$

$$a_0 = \frac{2}{l}\int_0^l f(x)dx = \frac{2}{\pi}\int_0^\pi x^2 dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^\pi = \frac{2}{\pi}\left[\frac{\pi^3}{3} - 0\right] = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$$

$\therefore$ Constant term $= \dfrac{a_0}{2} = \dfrac{\pi^2}{3}$

6. **Find the root mean square value of the function *f(x) = x* in *(0,l)***

RMS value = $\bar{y} = \sqrt{\dfrac{1}{b-a}\displaystyle\int_a^b [f(x)]^2 dx} = \sqrt{\dfrac{1}{l-0}\displaystyle\int_0^l x^2 dx} = \sqrt{\dfrac{1}{l}\left[\dfrac{x^3}{3}\right]_0^l} = \sqrt{\dfrac{l^3}{3l}} = \dfrac{l}{\sqrt{3}}$

7. **What do you mean by Harmonic analysis ?**

The process of finding the harmonics in the Fourier series expansion of a function numerically is known as harmonic analysis.

8. **Find the constant term in the expression of $\cos^2 x$ as a Fourier series in the interval $(-\pi,\pi)$.**

Solution:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2 x dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx=\frac{1}{2\pi}\left[x+\frac{\sin 2x}{2}\right]_{-\pi}^{\pi}\quad a_0=\frac{1}{2\pi}\left[(\pi)-(-\pi)\right]=\frac{2\pi}{2\pi}=1$$

$\therefore$ **Constant term =** $\boxed{\frac{a_0}{2}=\frac{1}{2}}$

PART B

1. **Find the Fourier series $f(x)=\left(\frac{\pi-x}{2}\right)^2$ in $0<x<2\pi$. Hence show that**

(i) $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+............=\frac{\pi^2}{6}$. **(ii)** $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+...=\frac{\pi^2}{12}$.

Solution:

Given $f(x)=\left(\frac{\pi-x}{2}\right)^2=\frac{1}{4}(\pi-x)^2=\frac{1}{4}\left(\pi^2-2\pi x+x^2\right),0<x<2\pi$

General Fourier is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{\ell}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}$

Here $\ell=\frac{\text{Upper Limit}-\text{Lower Limit}}{2}=\frac{2\pi-0}{2}=\frac{2\pi}{2}=\pi \qquad \therefore\ \ell=\pi$

$$\boxed{f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx}\quad ------(1)$$

To Find a_0:

$$a_0=\frac{1}{\ell}\int_0^{2\ell}f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}\left(\pi^2-2\pi x+x^2\right)dx=\frac{1}{4\pi}\int_0^{2\pi}\left(\pi^2-2\pi x+x^2\right)dx$$

$$=\frac{1}{4\pi}\left[\pi^2 x-\frac{2\pi x^2}{2}+\frac{x^3}{3}\right]_0^{2\pi}=\frac{1}{4\pi}\left[\left(2\pi^3-4\pi^3+\frac{8\pi^3}{3}\right)-(0)\right]$$

$$=\frac{\pi^3}{4\pi}\left[-2+\frac{8}{3}\right]=\frac{\pi^2}{4}\left[\frac{-6+8}{3}\right]=\frac{\pi^2}{4}\left[\frac{2}{3}\right]$$

$$\boxed{a_0=\frac{\pi^2}{6}}$$

To Find a_n

$$a_n=\frac{1}{\ell}\int_0^{2\ell}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}\left(\pi^2-2\pi x+x^2\right)\cos nx\,dx=\frac{1}{4\pi}\int_0^{2\pi}\left(\pi^2-2\pi x+x^2\right)\cos nx\,dx$$

$$=\frac{1}{4\pi}\left[\left(\pi^2-2\pi x+x^2\right)\left(\frac{\sin nx}{n}\right)-\left(-2\pi+2x\right)\left(\frac{-\cos nx}{n^2}\right)+(2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$=\frac{1}{4\pi}\left[\frac{1}{n^2}\left(-2\pi+2x\right)\cos nx\right]_0^{2\pi}$$

$$=\frac{1}{4\pi n^2}\left[\left(2\pi\cos 2n\pi\right)-\left(-2\pi\cos 0\right)\right]$$

$$a_n=\frac{1}{4\pi n^2}\left[2\pi+2\pi\right]=\frac{4\pi}{4\pi n^2}\qquad\because\cos 2n\pi=1\ \&\ \cos 0=1$$

$$\boxed{a_n = \frac{1}{n^2}}$$

To find b_n:

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{\ell}\int_0^{2\ell} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{4}\left(\pi^2-2\pi x+x^2\right)\sin nx\,dx = \frac{1}{4\pi}\int_0^{2\pi}\left(\pi^2-2\pi x+x^2\right)\sin nx\,dx$$

$$= \frac{1}{4\pi}\left[\left(\pi^2-2\pi x+x^2\right)\left(\frac{-\cos nx}{n}\right)-\left(-2\pi+2x\right)\left(\frac{-\sin nx}{n^2}\right)+(2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\frac{-1}{n}\left(\pi^2-2\pi x+x^2\right)\cos nx+\frac{2}{n^3}\cos nx\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(\frac{-1}{n}\left(\pi^2-4\pi^2+4\pi^2\right)\cos 2n\pi+\frac{2}{n^3}\cos 2n\pi\right)-\left(\frac{-1}{n}\pi^2\cos 0+\frac{2}{n^3}\cos 0\right)\right]$$

$$= \frac{1}{4\pi}\left(\frac{-\pi^2}{n}+\frac{-\pi^2}{n}+\frac{2}{n^3}-\frac{2}{n^3}\right)$$

$$\boxed{b_n = 0}$$

Substitute a_0, a_n, b_n in (1)

$$f(x) = \frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx+\sum_{n=1}^{\infty}0\sin nx$$

$$\therefore \boxed{f(x) = \frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx} \text{------}(2)$$

Deduction:

(i) Let $x=0$ be a point of discontinuity

$$(2) \Rightarrow \quad f(0) = \frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}\cos 0 \qquad \because f(0) = \frac{f(0)+f(2\pi)}{2} = \frac{\frac{\pi^2}{4}+\frac{\pi^2}{4}}{2} = \frac{2\frac{\pi^2}{4}}{2} = \frac{\pi^2}{4}$$

$$\because f(x) = \left(\frac{\pi-x}{2}\right)^2$$

$$\frac{\pi^2}{4} = \frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{4}-\frac{\pi^2}{12} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{4\pi^2}{48} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \Rightarrow \boxed{\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots\ldots\ldots = \frac{\pi^2}{6}}$$

ii) Let $x=\pi$ be a point of continuity

$$f(\pi) = \frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi \qquad \because f(x) = \left(\frac{\pi-x}{2}\right)^2 \Rightarrow f(0) = \left(\frac{\pi-\pi}{2}\right)^2 = 0$$

$$0=\frac{\pi^2}{12}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12}=\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12}=\frac{-1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}+\frac{1}{4^2}-...$$

$$\boxed{\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+...=\frac{\pi^2}{12}}$$

2. **Obtain the Fourier series for *f(x)* of period *2l* and defined as follows** $f(x)=\begin{cases} l-x & , \ 0<x\le l \\ 0 & , \ l\le x<2l \end{cases}$**. Hence deduce that (i)** $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...=\frac{\pi}{4}$ **(ii)** $1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+...=\frac{\pi^2}{8}$

Solution: General Fourier is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{\ell}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}$

Here $\ell=\frac{\text{Upper Limit}-\text{Lower Limit}}{2}=\frac{l-(-l)}{2}=\frac{2l}{2}=l \quad \therefore \ \ell=l$

To find a_0

$$a_0=\frac{1}{\ell}\int_0^{2\ell}f(x)\,dx=\frac{1}{\ell}\left[\int_0^{\ell}(\ell-x)\,dx+\int_{\ell}^{2\ell}0\,dx\right]$$

$$=\frac{1}{\ell}\left[lx-\frac{x^2}{2}\right]_0^{\ell}=\frac{1}{\ell}\left[l^2-\frac{l^2}{2}\right]=\frac{1}{\ell}\left[\frac{l^2}{2}\right]$$

$$\boxed{a_0=\frac{\ell}{2}}$$

$$a_n=\frac{1}{\ell}\int_0^{2\ell}f(x)\cos\frac{n\pi x}{\ell}\,dx=\frac{1}{\ell}\left[\int_0^{\ell}(\ell-x)\cos\frac{n\pi x}{\ell}\,dx+\int_{\ell}^{2\ell}0\,dx\right]$$

$$=\frac{1}{\ell}\left[(\ell-x)\left(\frac{\sin\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right)-(-1)\left(\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n^2\pi^2}{\ell^2}}\right)\right]_0^{\ell}$$

$$=\frac{1}{\ell}\left[\frac{-l^2}{n^2\pi^2}\cos\frac{n\pi x}{\ell}\right]_0^{l}$$

$$=\frac{-\ell^2}{l\,n^2\pi^2}\left[\cos n\pi-\cos 0\right]=\frac{\ell}{n^2\pi^2}\left[1-(-1)^n\right]$$

$$\boxed{a_n=\begin{cases}\frac{2\ell}{n^2\pi^2}, & n=1,3,5,... \\ 0 & , n=2,4,6,...\end{cases}}$$

$$b_n=\frac{1}{\ell}\int_0^{2\ell}f(x)\sin\frac{n\pi x}{\ell}\,dx=\frac{1}{\ell}\left[\int_0^{\ell}(\ell-x)\sin\frac{n\pi x}{\ell}\,dx+\int_{\ell}^{2\ell}0\,dx\right]$$

$$b_n=\frac{1}{\ell}\left[(\ell-x)\left(\frac{-\cos\frac{n\pi x}{\ell}}{\frac{n\pi}{\ell}}\right)-(-1)\left(\frac{-\sin\frac{n\pi x}{\ell}}{\frac{n^2\pi^2}{\ell^2}}\right)\right]_0^{\ell}$$

$$=\frac{1}{\ell}\left[\frac{-l}{n\pi}(l-x)\cos\frac{n\pi x}{\ell}\right]_0^{l}=\frac{-l}{\ell n\pi}\left[0-l\cos 0\right]=\frac{l^2}{ln\pi}$$

$$\boxed{b_n=\frac{\ell}{n\pi}}$$

Substitute a_0, a_n, b_n in (1)

$$f(x)=\frac{\ell}{4}+\sum_{n=1,3,5}^{\infty}\frac{2\ell}{n^2\pi^2}\cos\frac{n\pi x}{\ell}+\sum_{n=1}^{\infty}\frac{\ell}{n\pi}\sin\frac{n\pi x}{\ell}$$

$$\therefore\ f(x)=\frac{\ell}{4}+\frac{2\ell}{\pi^2}\sum_{n=1,3,5...}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{\ell}+\frac{\ell}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{\ell}\quad ------(2)$$

Deduction:

i) Let $x=\ell/2$ be a point of continuity

$$(2)\Rightarrow f(\ell/2)=\frac{\ell}{4}+\frac{2\ell}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{2}+\frac{\ell}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\qquad \because f(x)=\begin{cases}l-x, & 0<x\le l\\ 0, & l\le x<2l\end{cases}$$

$$\Rightarrow\frac{\ell}{2}=\frac{\ell}{4}+\frac{2\ell}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\cos\frac{n\pi}{2}+\frac{\ell}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\qquad \because f\left(\frac{l}{2}\right)=l-\frac{l}{2}=\frac{l}{2}$$

$$\Rightarrow\ \frac{\ell}{2}-\frac{\ell}{4}=0+\frac{\ell}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\qquad \because \cos\frac{n\pi}{2}=0\ \text{if n is odd}$$

$$\Rightarrow\ \frac{\ell}{4}=\frac{\ell}{\pi}\left[\frac{1}{1}\sin\frac{\pi}{2}+\frac{1}{2}\sin\pi+\frac{1}{3}\sin\frac{3\pi}{2}+...\right]$$

$$\Rightarrow\ \frac{\pi}{4}=\frac{1}{1}(1)+\frac{1}{2}(0)+\frac{1}{3}(-1)+....\qquad \because \sin\frac{\pi}{2}=1;\sin\frac{3\pi}{2}=-1;\ \sin\frac{5\pi}{2}=1;\sin\frac{7\pi}{2}=-1\ \text{etc..}$$

$$\boxed{1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...=\frac{\pi}{4}}$$

i) Let $x=\ell$ be a point of continuity

$$(2)\Rightarrow\ f(l)=\frac{\ell}{4}+\frac{2\ell}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\cos n\pi+\frac{\ell}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin n\pi\qquad \because f(x)=\begin{cases}l-x, & 0<x\le l\\ 0, & l\le x<2l\end{cases}$$

$$0=\frac{\ell}{4}+\frac{2\ell}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}(-1)^n+0\qquad \Rightarrow f(l)=l-l=0$$

$$-\frac{\ell}{4}=\frac{2\ell}{\pi^2}\left[-\frac{1}{1^2}+\frac{1}{3^2}-\frac{1}{5^2}+...\right]$$

$$-\frac{\pi^2}{8}=-\left[\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-...\right]$$

$$\boxed{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-...=\frac{\pi^2}{8}}$$

3. **Find the Fourier series expansion of** $f(x)=\begin{cases}x, & 0\le x\le 3\\ 6-x, & 3\le x\le 6\end{cases}$ **and hence find the value**

of $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+.......$

Solution:

Given $f(x)=\begin{cases} x, & 0\le x\le 3 \\ 6-x, & 3\le x\le 6 \end{cases}$

General Fourier is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$

Here $\ell=\frac{\text{Upper Limit}-\text{Lower Limit}}{2}=\frac{6-0}{2}=3 \qquad \therefore\ \ell=3$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{3}+\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{3} \qquad -------(1)$$

$$a_0=\frac{1}{\ell}\int_0^{2\ell} f(x)\,dx=\frac{1}{3}\int_0^{6} f(x)\,dx=\frac{1}{3}\left[\int_0^3 x\,dx+\int_3^6 (6-x)\,dx\right]$$

$$=\frac{1}{3}\left\{\left[\frac{x^2}{2}\right]_0^3+\left[\frac{(6-x)^2}{-2}\right]_3^6\right\}$$

$$=\frac{1}{3}\left\{\left[\left\{\frac{3^2}{2}\right\}-\{0\}\right]+\left[\{0\}-\left\{\frac{(3)^2}{-2}\right\}\right]\right\}=\frac{1}{3}\left[\frac{9}{2}+\frac{9}{2}\right]=\frac{9}{3}$$

$$\boxed{a_0=3}$$

$$a_n=\frac{1}{\ell}\int_0^{2\ell} f(x)\cos\frac{n\pi x}{\ell}dx$$

$$=\frac{1}{3}\int_0^{6} f(x)\cos\frac{n\pi x}{3}dx$$

$$=\frac{1}{3}\left[\int_0^3 x\cos\frac{n\pi x}{3}dx+\int_3^6 (6-x)\cos\frac{n\pi x}{3}dx\right]$$

$$=\frac{1}{3}\left\{\left[(x)\left(\frac{\sin\frac{n\pi x}{3}}{\frac{n\pi}{3}}\right)-(1)\left(\frac{-\cos\frac{n\pi x}{3}}{\frac{n^2\pi^2}{9}}\right)\right]_0^3+\left[(6-x)\left(\frac{\sin\frac{n\pi x}{3}}{\frac{n\pi}{3}}\right)-(-1)\left(\frac{-\cos\frac{n\pi x}{3}}{\frac{n^2\pi^2}{9}}\right)\right]_3^6\right\}$$

$$=\frac{1}{3}\left\{\left[\frac{9}{n^2\pi^2}\cos\frac{n\pi x}{3}\right]_0^3-\left[\frac{9}{n^2\pi^2}\cos\frac{n\pi x}{3}\right]_3^6\right\}$$

$$=\frac{9}{3n^2\pi^2}\left\{\left[\cos\frac{n\pi x}{3}\right]_0^3-\left[\cos\frac{n\pi x}{3}\right]_3^6\right\}$$

$$=\frac{3}{n^2\pi^2}\left\{\left[\cos n\pi-\cos 0\right]-\left[\cos 2n\pi-\cos n\pi\right]\right\}$$

$$=\frac{3}{n^2\pi^2}\left[\cos n\pi-1-1+\cos n\pi\right]=\frac{3}{n^2\pi^2}\left[2(-1)^n-2\right]=\frac{6}{n^2\pi^2}\left[(-1)^n-1\right]$$

$$=\frac{6}{n^2\pi^2}\begin{cases} -2, & n=1,3,5,... \\ 0, & n=2,4,6,... \end{cases}$$

$$a_n = \begin{cases} \dfrac{-12}{n^2\pi^2}, & n=1,3,5,\ldots \\ 0, & n=2,4,6,\ldots \end{cases}$$

$$b_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx = \frac{1}{3}\int_0^{6} f(x)\sin\frac{n\pi x}{3}\,dx = \frac{1}{3}\left[\int_0^{3} x\sin\frac{n\pi x}{3}\,dx + \int_3^{6}(6-x)\sin\frac{n\pi x}{3}\,dx\right]$$

$$= \frac{1}{3}\left\{\left[(x)\left(\frac{-\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}}\right) - (1)\left(\frac{-\sin\frac{n\pi x}{3}}{\frac{n^2\pi^2}{9}}\right)\right]_0^3 + \left[(6-x)\left(\frac{-\cos\frac{n\pi x}{3}}{\frac{n\pi}{3}}\right) - (-1)\left(\frac{-\sin\frac{n\pi x}{3}}{\frac{n^2\pi^2}{9}}\right)\right]_3^6\right\}$$

$$= \frac{1}{3}\left\{\left[\frac{-3}{n\pi}x\cos\frac{n\pi x}{3}\right]_0^3 + \left[\frac{-3}{n\pi}(6-x)\cos\frac{n\pi x}{3}\right]_3^6\right\}$$

$$= \frac{-3}{3n\pi}\left\{\left[3\cos n\pi - 0\right] + \left[0 - 3\cos n\pi\right]\right\} = \frac{-3}{3n\pi}\left[3\cos n\pi - 3\cos n\pi\right]$$

$$\therefore \boxed{b_n = 0}$$

Substitute a_0, a_n, b_n in (1)

$$f(x) = \frac{3}{2} + \sum_{n=1,3,5}^{\infty} \frac{-12}{n^2\pi^2}\cos\frac{n\pi x}{3} + \sum_{n=1}^{\infty} 0\,\sin\frac{n\pi x}{3}$$

$$\therefore \boxed{f(x) = \frac{3}{2} - \frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\frac{n\pi x}{3}} \text{------}(2)$$

Deduction:

Let $x = 0$ be a point of continuity

$$f(0) = \frac{3}{2} - \frac{12}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\cos 0 \qquad f(x) = \begin{cases} x, & 0 \le x \le 3 \\ 6-x, & 3 \le x \le 6 \end{cases}$$

$$\Rightarrow 0 = \frac{3}{2} - \frac{12}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \frac{12}{\pi^2}\sum_{n=1,3,5}^{\infty}\frac{1}{n^2} = \frac{3}{2}$$

$$\therefore \sum_{n=1,3,5}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8} \qquad \Rightarrow \boxed{\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}}$$

4. **Find the Fourier series for $f(x) = |x|$ in $-\pi < x < \pi$ and deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{8}$.**

Solution:

Given $f(x) = |x|, \quad -\pi < x < \pi$

Here $\ell = \dfrac{\text{Upper Limit} - \text{Lower Limit}}{2} = \dfrac{\pi - (-\pi)}{2} = \dfrac{2\pi}{2} \qquad \therefore \ell = \pi$

Now, $f(-x) = |-x| = |x| = f(x)$

$\therefore f(x)$ is an even function.

$\therefore \boxed{b_n = 0}$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}$$

$$\therefore\ f(x)=\frac{a_0}{2}+\sum_{1}^{\infty} a_n \cos nx \ \ ------(1) \qquad \because l=\pi$$

To Find a_0:

$$a_0=\frac{2}{l}\int_0^l f(x)\,dx=\frac{2}{\pi}\int_0^\pi |x|\,dx=\frac{2}{\pi}\int_0^\pi x\,dx$$

$$=\frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^\pi=\frac{2}{2\pi}\left(\pi^2-0\right)=\pi$$

$$\therefore\ \boxed{a_0=\pi}$$

To Find a_n:

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx=\frac{2}{\pi}\int_0^\pi |x|\cos nx\,dx=\frac{2}{\pi}\int_0^\pi x\cos nx\,dx$$

$$=\frac{2}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right)-(1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^\pi$$

$$=\frac{2}{\pi}\left[\left(\frac{1}{n^2}\right)\cos nx\right]_0^\pi$$

$$=\frac{2}{\pi}\left(\frac{1}{n^2}\right)\left[\cos n\pi-\cos 0\right]$$

$$=\frac{2}{\pi n^2}\left[(-1)^n-1\right]$$

$$\boxed{a_n=\begin{cases}\dfrac{-4}{\pi n^2}, & if\ n=1,3,5,\ldots\\ 0, & if\ n=2,4,6,\ldots\end{cases}}$$

$$\therefore\ (1)\Rightarrow f(x)=\frac{\pi}{2}+\sum_{n=1,3,5\ldots}^{\infty}\frac{-4}{n^2\pi}\cos nx$$

$$\therefore\ \boxed{f(x)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\cos nx} \ \ --------(2)$$

Deduction:

Let $x=0$ be a point of continuity

$$(2)\ \Rightarrow f(0)=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\cos 0$$

$$\Rightarrow 0=\frac{\pi}{2}-\frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2} \qquad f(x)=|x|\ \Rightarrow f(0)=0$$

$$\Rightarrow \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}=\frac{\pi}{2}$$

$$\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{8}$$

$$\therefore \boxed{1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots=\frac{\pi^2}{8}}$$

5. Find the Fourier series for $f(x) = x^2$ in $-\pi \le x \le \pi$ and deduce that

(a) $1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$.

(b) $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}=\frac{\pi^2}{12}$.

(c) $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots=\frac{\pi^4}{90}$

Solution:

Given $f(x)=x^2,\ -\pi \le x \le \pi$

Here $\ell=\frac{\text{Upper Limit}-\text{Lower Limit}}{2}=\frac{\pi-(-\pi)}{2}=\frac{2\pi}{2}=\pi \qquad \therefore\ \ell=\pi$

Now, $f(-x)=x^2=(-x)^2=x^2=f(x)$

$\therefore\ f(x)$ is an even function.

$\therefore b_n=0$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}$$

$$\therefore f(x)=\frac{a_0}{2}+\sum_{1}^{\infty} a_n \cos nx \quad ------(1) \qquad \because l=\pi$$

To Find a_0:

$$a_0=\frac{2}{l}\int_0^l f(x)\,dx=\frac{2}{\pi}\int_0^{\pi} x^2\,dx=\frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}=\frac{2}{3\pi}\left[\pi^3-0\right]=\frac{2\pi^2}{3}$$

$$\boxed{a_0=\frac{2\pi^2}{3}}$$

To find a_n:

$$a_n=\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx=\frac{2}{\pi}\int_0^{\pi} x^2\cos\ nx\,dx$$

$$=\frac{2}{\pi}\left[(x^2)\left(\frac{\sin nx}{n}\right)-(2x)\left(\frac{-\cos nx}{n^2}\right)+(2)\left(\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\left(\frac{2}{n^2}\right)x\cos nx\right]_0^{\pi}$$

$$=\frac{4}{\pi n^2}(\pi\cos n\pi-0)$$

$$\boxed{a_n=\frac{4(-1)^n}{n^2}}$$

$$\therefore (1) \Rightarrow f(x)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

$$\therefore \boxed{f(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx} \text{-------} (2)$$

Deduction: a

Let $x=\pi$ be a point of continuity

$$(2) \Rightarrow f(\pi)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)}{n^2}\cos\pi$$

$$\Rightarrow \pi^2=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{n^2} \qquad \because f(x)=x^2 \Rightarrow f(\pi)=\pi^2$$

$$\Rightarrow \pi^2-\frac{\pi^2}{3}=4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$

$$\Rightarrow \frac{2\pi^2}{12}=\sum_{n=1}^{\infty}\frac{1}{n^2} \qquad \because (-1)^{2n}=1$$

$$\therefore \boxed{1+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}}$$

Deduction: b

Let $x=0$ be a point of continuity

$$(2) \Rightarrow f(0)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)}{n^2}\cos 0$$

$$\Rightarrow 0=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \qquad \because f(x)=x^2 \Rightarrow f(0)=0$$

$$\Rightarrow -\frac{\pi^2}{3}=4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\Rightarrow -\frac{1}{1^2}+\frac{1}{2^2}-\frac{1}{3^2}=-\frac{\pi^2}{12}$$

$$\Rightarrow \boxed{\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}=\frac{\pi^2}{12}}$$

Deduction: c

By Parsevals identity for Fourier series,

$$\frac{2}{l}\int_0^l\left[f(x)\right]^2dx=\frac{a_0{}^2}{2}+\sum_{n=1}^{\infty}a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi}\left[x^2\right]^2dx=\frac{\frac{4\pi^4}{9}}{2}+\sum_{n=1}^{\infty}\left(\frac{4(-1)^n}{n^2}\right)^2$$

$$\frac{2}{\pi}\int_0^{\pi}x^4dx=\frac{2\pi^4}{9}+16\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^4}$$

$$\frac{2}{\pi}\left[\left(\frac{\pi^5}{5}\right)-(0)\right]=\frac{2\pi^4}{9}+16\sum_{n=1}^{\infty}\frac{1}{n^4} \qquad \because (-1)^{2n}=1$$

$$\frac{2\pi^5}{5\pi}=\frac{2\pi^4}{9}+16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{2\pi^4}{5}-\frac{2\pi^4}{9}=16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$2\pi^4\left(\frac{1}{5}-\frac{1}{9}\right)=16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$2\pi^4\times\frac{4}{45}=16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{8\pi^4}{45\times 16}=\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{90}=\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots=\frac{\pi^4}{90}$$

6. Obtain the Fourier series of $f(x) = x + x^2$ $in -\pi < x < \pi$. Hence show that

$i)\ \sum_{1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$ $ii)\ \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\dots=\frac{\pi^2}{12}$

Solution:

Given $f(x)=x+x^2$ **in** $-\pi\le x\le\pi$

Here $\ell=\frac{\text{Upper Limit}-\text{Lower Limit}}{2}=\frac{\pi-(-\pi)}{2}=\frac{2\pi}{2}=\pi$ $\quad\therefore\ \ell=\pi$

Now, $f(-x)=-x+(-x)^2$

$=-x+x^2\ne f(x)$

$=-(x-x^2)\ne -f(x)$

$\therefore f(-x)\ne f(x)$ & $f(-x)\ne -f(x)$

$\therefore\ f(x)$ is Neithereven Nor odd function.

General Fourier is $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{\ell}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}$

$\therefore f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$ ------- (1) $\because l=\pi$

To Find a_0:

$$a_0=\frac{1}{\ell}\int_{-l}^{l}f(x)\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)dx=\frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$=\frac{1}{\pi}\left\{\left(\frac{\pi^2}{2}+\frac{\pi^3}{3}\right)-\left(\frac{\pi^2}{2}-\frac{\pi^3}{3}\right)\right\}=\frac{1}{\pi}\left\{\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right\}=\frac{1}{\pi}\left\{\frac{2\pi^3}{3}\right\}=\frac{2\pi^2}{3}$$

$$\boxed{a_0=\frac{2\pi^2}{3}}$$

To Find a_n:

$$a_n=\frac{1}{\ell}\int_{-l}^{l}f(x)\cos\frac{n\pi x}{l}\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left[\left(x+x^2\right)\left(\frac{\sin nx}{n}\right) - (1+2x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{1}{n^2}\right)(1+2x)\cos nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi n^2}\left\{\left[(1+2\pi)\cos n\pi -\right] - \left[(1-2\pi)\cos(-n\pi)\right]\right\}$$

$$= \frac{1}{\pi n^2}\left[(-1)^n\left(1+2\pi-1+2\pi\right)\right] \qquad \because \cos(-n\pi)=\cos n\pi = (-1)^n$$

$$= \frac{(-1)^n}{n^2\pi}[4\pi] = \frac{4(-1)^n}{n^2}$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

To Find b$_n$:

$$b_n = \frac{1}{\ell}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x+x^2\right)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\left(x+x^2\right)\left(\frac{-\cos nx}{n}\right) - (1+2x)\left(\frac{-\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{-1}{n}\right)\left(x+x^2\right)\cos nx + \left(\frac{2}{n^3}\right)\cos nx\right]_{-\pi}^{\pi} \qquad \because \cos(-n\pi)=\cos n\pi=(-1)^n$$

$$= \frac{1}{\pi}\left\{\left[\left(\frac{-1}{n}\right)\left(\pi+\pi^2\right)\cos n\pi + \left(\frac{2}{n^3}\right)\cos n\pi\right] - \left[\left(\frac{-1}{n}\right)\left(-\pi+\pi^2\right)\cos n\pi + \left(\frac{2}{n^3}\right)\cos(-n\pi)\right]\right\}$$

$$= \frac{(-1)^n}{\pi}\left(-\frac{\pi}{n} - \frac{\pi^2}{n} + \frac{2}{n^3} - \frac{\pi}{n} + \frac{\pi^2}{n} - \frac{2}{n^3}\right) = -\frac{2\pi(-1)^n}{n\pi}$$

$$\therefore \boxed{b_n = \frac{-2}{n}(-1)^n}$$

$$(1) \Rightarrow f(x) = \frac{\pi^2}{3} + \sum_{1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx + \sum_{1}^{\infty}\frac{-2(-1)^n}{n}\sin nx$$

$$\boxed{f(x) = \frac{\pi^2}{3} + 4\sum_{1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{1}^{\infty}\frac{(-1)^n}{n}\sin nx} \quad -----(2)$$

Deduction: 1

Let $x=\pi$ be the point of discontinuity

$$(2) \Rightarrow f(\pi) = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}\cos n\pi - 2\sum_1^\infty \frac{(-1)^n}{n}\sin n\pi$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{1}{n^2} \quad \because f(x) = x + x^2 \Rightarrow f(\pi) = \frac{f(-\pi)+f(\pi)}{2} = \frac{-\pi+\pi^2+\pi+\pi^2}{2} = \frac{2\pi^2}{2} = \pi^2$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_1^\infty \frac{1}{n^2}$$

$$\frac{2\pi^2}{3\times 4} = \sum_1^\infty \frac{1}{n^2}$$

$$\boxed{\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}}$$

Deduction: 2

Let $x = 0$ be the point of continuity

$$(2) \Rightarrow f(0) = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}\cos 0 - 2\sum_1^\infty \frac{(-1)^n}{n}\sin 0$$

$$0 = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2} \qquad f(x) = x + x^2 \Rightarrow f(0) = 0$$

$$-\frac{\pi^2}{3} = 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^3}\ldots\right]$$

$$\boxed{\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^3} - \ldots = \frac{\pi^2}{12}}$$

7. Find the Fourier series expansion of f (x) where $f(x) = \begin{cases} \pi + x, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}$ and hence deduce that

$$\sum_1^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

Solution:

Given $f(x) = \begin{cases} \pi + x, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi \end{cases}$

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} = \pi \qquad \therefore \ell = \pi$

Let $f(x) = \begin{cases} f_1(x), & -\pi \le x \le 0 \\ f_2(x), & 0 \le x \le \pi \end{cases}$

Where

$f_1(x) = \pi + x \qquad\qquad f_2(x) = \pi - x$

$f_1(-x) = \pi - x = f_2(x) \qquad f_2(-x) = \pi + x = f_1(x)$

$\therefore f(x)$ is aneven function.

$\therefore \boxed{b_n = 0}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos\frac{n\pi x}{\ell}$$

$$\therefore \boxed{f(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos nx} \; ------(1) \quad \because l = \pi$$

To Find a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(\pi - x)\ dx = \frac{2}{\pi}\int_0^{\pi} x\ dx$$

$$= \frac{2}{\pi}\left(\pi x - \frac{x^2}{2}\right)_0^{\pi} = \frac{2}{\pi}\left(\pi^2 - \frac{\pi^2}{2}\right) = \frac{2}{\pi}\times\frac{\pi^2}{2} = \pi$$

$$\therefore \boxed{a_0 = \pi}$$

To Find a_n:

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi - x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{-1}{n^2}\right)\cos nx\right]_0^{\pi}$$

$$= \frac{-2}{n^2\pi}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{-2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$\boxed{a_n = \begin{cases}\dfrac{4}{\pi n^2}, & if\ n = 1,3,5,\ldots \\ 0, & if\ n = 2,4,6,\ldots\end{cases}}$$

$$\therefore (1) \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1,3,5\ldots}^{\infty}\frac{4}{n^2\pi}\cos nx$$

$$\therefore \boxed{f(x) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\cos nx} \quad --------(2)$$

Deduction:

Let $x = 0$ be a point of continuity

$$(2) \Rightarrow f(0) = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\cos 0$$

$$\Rightarrow \pi = \frac{\pi}{2} + \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2} \qquad \because f(x) = \begin{cases}\pi + x, & -\pi \le x \le 0 \\ \pi - x, & 0 \le x \le \pi\end{cases} \Rightarrow f(0) = \pi$$

$$\Rightarrow \frac{4}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2} = \pi - \frac{\pi}{2}$$

$$\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2} = \frac{\pi}{2}\times\frac{\pi}{4} = \frac{\pi^2}{8}$$

$$\therefore \boxed{\sum_{1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.} \qquad \because \sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n} = \sum_{n=1}^{\infty}\frac{1}{2n-1}$$

8. Determine the Fourier series expansion of f (x) where $f\left(x\right)=\begin{cases}-1+x, & -\pi\le x\le 0\\ 1+x, & 0\le x\le\pi\end{cases}$ with $f(x+2\pi)=f(x)$.

Solution:

Given $f\left(x\right)=\begin{cases}-1+x, & -\pi\le x\le 0\\ 1+x, & 0\le x\le\pi\end{cases}$

Here $\ell=\dfrac{\text{Upper Limit}-\text{Lower Limit}}{2}=\dfrac{\pi-(-\pi)}{2}=\dfrac{2\pi}{2}=\pi \qquad \therefore\ \ell=\pi$

Let $f\left(x\right)=\begin{cases}f_1(x), & -\pi\le x\le 0\\ f_2(x), & 0\le x\le\pi\end{cases}$

Where

$f_1(x)=-1+x \qquad\qquad f_2(x)=1+x$

$f_1(-x)=-1-x=-(1+x)=-f_2(x) \qquad\qquad f_2(-x)=1-x=-(-1+x)=-f_1(x)$

$\therefore\ f(x)$ is an odd function.

$\therefore\ \boxed{a_0=a_n=0}$

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}$$

$$\therefore\ f\left(x\right)=\sum_{1}^{\infty}b_n\sin nx\ \ ------(1) \qquad \because l=\pi$$

To Find b_n:

$$b_n=\frac{1}{l}\int_{-l}^{l}f(x)\sin\frac{n\pi x}{l}dx=\frac{2}{\pi}\int_{0}^{\pi}(1+x)\sin nxdx$$

$$=\frac{2}{\pi}\left[(1+x)\left(\frac{-\cos nx}{n}\right)-(1)\cancel{\left(\frac{-\sin nx}{n^2}\right)}\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\left(\frac{-1}{n}\right)(1+x)\cos nx\right]_0^{\pi}$$

$$=\frac{-2}{n\pi}\left[(1+\pi)\cos n\pi-(1+0)\cos 0\right]$$

$$=\frac{-2}{n\pi}\left[(1+\pi)(-1)^n-1\right]$$

$$\boxed{b_n=\frac{2}{n\pi}\left[1-(1+\pi)(-1)^n\right]}$$

$$\therefore\ (1)\Rightarrow f\left(x\right)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\left[1-(1+\pi)(-1)^n\right]\sin nx$$

$$\therefore\ \boxed{f\left(x\right)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1-(1+\pi)(-1)^n\right]\sin nx}$$

9. Find the Fourier series for f (x) where $f\left(x\right)=\begin{cases}0, & -1<x<0\\ 1, & 0<x<1\end{cases}$.

Solution:

Given $f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 < x < 1 \end{cases}$

Here $\ell = \dfrac{\text{Upper Limit} - \text{Lower Limit}}{2} = \dfrac{1-(-1)}{2} = \dfrac{2}{2} = 1 \qquad \therefore\ \ell = 1$

Let $f(x) = \begin{cases} f_1(x), & -1 \le x \le 0 \\ f_2(x), & 0 \le x \le 1 \end{cases}$

Where

$f_1(x) = 0 \qquad f_2(x) = 1$

$f_1(-x) = 0 \ne f_2(x) \qquad f_2(-x) = 1 \ne f_1(x)$

$f_1(-x) = 0 \ne -f_2(x) \qquad f_2(-x) = 1 \ne -f_1(x)$

$\therefore\ f(x)$ is Neither even Nor odd function.

General Fourier is $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\dfrac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\dfrac{n\pi x}{\ell}$

$$\therefore \boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x} \text{ ------- } (1) \quad \because l = 1$$

To Find a_0:

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 0\,dx + \int_{0}^{1} 1\,dx = [x]_0^1 = 1 - 0 = 1$$

$$\boxed{\therefore\ a_0 = 1}$$

To Find a_n:

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{1}\int_{-1}^{1} f(x)\cos\frac{n\pi x}{1}\,dx$$

$$= \int_{-1}^{0} 0\,dx + \int_{0}^{1} 1\cos n\pi x\,dx$$

$$= \left[\frac{\sin n\pi x}{n\pi}\right]_0^1$$

$$= [\sin n\pi - \sin 0] = 0$$

$$\boxed{a_n = 0}$$

To Find b_n:

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{1}\int_{-1}^{1} f(x)\sin\frac{n\pi x}{1}\,dx$$

$$= \int_{-1}^{0} 0\,dx + \int_{0}^{1} 1\sin n\pi x\,dx$$

$$= \left[\frac{-\cos n\pi x}{n\pi}\right]_0^1$$

$$= \frac{-1}{n\pi}\left[\cos n\pi x\right]_0^1$$

$$= \frac{-1}{n\pi}\left[\cos n\pi - \cos 0\right]_0^1$$

$$= \frac{1}{n\pi}\left[1-(-1)^n\right]$$

$$\boxed{b_n = \begin{cases} \frac{2}{n\pi}, & if \quad n = 1,3,5,... \\ 0, & if \quad n = 2,4,6.... \end{cases}}$$

$$(1) \Rightarrow f(x) = \frac{1}{2} + \sum_{1,3}^{\infty} \frac{2}{n\pi} \sin n\pi x.$$

$$\boxed{f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{1,3}^{\infty} \frac{1}{n} \sin n\pi x}$$

10. **Find the Fourier series for $f(x) = |\cos x|$ in the interval (-π,π).**

Solution:

Given $f(x) = |\cos x|, \quad -\pi < x < \pi$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} \qquad \therefore \ \ell = \pi$$

Now, $f(-x) = |\cos(-x)| = |\cos x| = f(x)$

$\therefore$ $f(x)$ is an even function.

$\therefore$ $\boxed{b_n = 0}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\ell}$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos nx \quad ------(1) \qquad \because l = \pi$$

To find a_0:

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x| \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \, dx + \int_{\pi/2}^{\pi} -\cos x \, dx\right] \quad \because |\cos x| = \begin{cases} \cos x, & 0 < x < \frac{\pi}{2} \\ -\cos x, & \frac{\pi}{2} < x < \pi \end{cases}$$

$$= \frac{2}{\pi}\left[(\sin x)_0^{\pi/2} - (\sin x)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left(\sin\frac{\pi}{2} - \sin 0\right) - \left(\sin \pi - \sin\frac{\pi}{2}\right)\right] = \frac{2}{\pi}(1-(-1))$$

$$\therefore \boxed{a_0 = \frac{4}{\pi}}$$

To find a_n:

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}dx = \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx$$

$$a_n = \frac{2}{\pi}\int_0^\pi |\cos x|\cos nx dx$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2}\cos x\cos nx dx + \int_{\pi/2}^{\pi} -\cos x\cos nx dx\right]$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2}\cos nx\cos x dx - \int_{\pi/2}^{\pi}\cos nx\cos x dx\right]$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A+B)+\cos(A-B)\right] \quad \textit{Here} \quad A = nx \quad B = x$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\pi/2}\frac{1}{2}\left[\cos(n+1)x+\cos(n-1)x\right]dx - \int_{\pi/2}^{\pi}\frac{1}{2}\left[\cos(n+1)x+\cos(n-1)x\right]dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2}\frac{1}{2}\left[\cos(n+1)x+\cos(n-1)x\right]dx - \int_{\pi/2}^{\pi}\left[\cos(n+1)x+\cos(n-1)x\right]dx\right]$$

$$= \frac{1}{\pi}\left\{\left[\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right]_0^{\pi/2} - \left[\frac{\sin(n+1)x}{n+1}+\frac{\sin(n-1)x}{n-1}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\left(\frac{\sin(n+1)\frac{\pi}{2}}{n+1}+\frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right)-(0)\right] - \frac{1}{\pi}\left[(0)-\left(\frac{\sin(n+1)\frac{\pi}{2}}{n+1}+\frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left(\frac{\sin\left(\frac{n\pi}{2}+\frac{\pi}{2}\right)}{n+1}+\frac{\sin\left(\frac{n\pi}{2}-\frac{\pi}{2}\right)}{n-1}+\frac{\sin\left(\frac{n\pi}{2}+\frac{\pi}{2}\right)}{n+1}+\frac{\sin\left(\frac{n\pi}{2}-\frac{\pi}{2}\right)}{n-1}\right)$$

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin\left(\frac{n\pi}{2}+\frac{\pi}{2}\right) = \sin\frac{n\pi}{2}\cos\frac{\pi}{2}+\cos\frac{n\pi}{2}\sin\frac{\pi}{2} = \cos\frac{n\pi}{2} \quad \because \cos\frac{\pi}{2}=0 \,\&\, \sin\frac{\pi}{2}=1$$

$$\sin\left(\frac{n\pi}{2}-\frac{\pi}{2}\right) = \sin\frac{n\pi}{2}\cos\frac{\pi}{2}-\cos\frac{n\pi}{2}\sin\frac{\pi}{2} = -\cos\frac{n\pi}{2} \quad \because \cos\frac{\pi}{2}=0 \,\&\, \sin\frac{\pi}{2}=1$$

$$a_n = \frac{1}{\pi}\left(\frac{\cos\frac{n\pi}{2}}{n+1}-\frac{\cos\frac{n\pi}{2}}{n-1}+\frac{\cos\frac{n\pi}{2}}{n+1}-\frac{\cos\frac{n\pi}{2}}{n-1}\right)$$

$$= \frac{2}{\pi}\cos\frac{n\pi}{2}\left(\frac{1}{n+1}-\frac{1}{n-1}\right)$$

$$= \frac{2}{\pi}\cos\frac{n\pi}{2}\left(\frac{n-1-n-1}{(n+1)(n-1)}\right)$$

$$a_n = \frac{2}{\pi}\cos\frac{n\pi}{2}\left(\frac{-2}{(n^2-1)}\right)$$

$$\boxed{a_n = \frac{-4}{\pi(n^2-1)}\cos\frac{n\pi}{2}}$$, Provided $n \neq 1$

When $n = 1$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos x dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos x dx + \int_{\pi/2}^{\pi} -\cos x \cos x dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x\, dx - \int_{\pi/2}^{\pi} \cos^2 x\, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \left(\frac{1+\cos 2x}{2}\right) dx - \int_{\pi/2}^{\pi} \left(\frac{1+\cos 2x}{2}\right) dx\right]$$

$$= \frac{2}{2\pi}\left[\int_0^{\pi/2} (1+\cos 2x)\, dx - \int_{\pi/2}^{\pi} (1+\cos 2x)\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(x + \frac{\sin 2x}{2}\right)_0^{\pi/2} - \left(x + \frac{\sin 2x}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left\{\left[\left(\frac{\pi}{2} + \frac{\sin \cancel{\pi}}{\cancel{2}}\right) - (0)\right] - \left[\left(\pi + \frac{\sin 2\cancel{\pi}}{\cancel{2}}\right) - \left(\frac{\pi}{2} + \frac{\sin \cancel{\pi}}{\cancel{2}}\right)\right]\right\}$$

$$a_1 = \frac{1}{\pi}\left[\frac{\pi}{2} - \pi + \frac{\pi}{2}\right]$$

$$\boxed{a_1 = 0}$$

$$\therefore f(x) = \frac{\frac{4}{\pi}}{2} + \sum_1^{\infty} \frac{-4}{\pi(n^2-1)}\cos\frac{n\pi}{2}\cos nx$$

$$\therefore \boxed{f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_1^{\infty} \frac{1}{(n^2-1)}\cos\frac{n\pi}{2}\cos nx}$$

11. Find the half range Fourier sine series for $f(x) = x(\pi - x)$ **in the interval** $(0, \pi)$ **and deduce that**

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots\infty$$

Solution:

Given $f(x) = x(\pi - x) = \pi x - x^2, \quad (0, \pi)$

: General Fourier is $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$

Here $\ell = \text{Upper Limit} - \text{Lower Limit} = \pi - 0 = \pi \quad \therefore\ \ell = \pi$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad -----(1)$$

To Find b$_n$:

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell} dx$$

$$= \frac{2}{\ell}\int_0^{\ell} x(\pi - x)\sin nx\, dx$$

$$= \frac{2}{\ell}\int_0^{\ell}(\pi x - x^2)\sin nx\,dx$$

$$b_n = \frac{2}{\pi}\left[(\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2}{n^3}\cos nx\right]_0^{\pi}$$

$$= \frac{-4}{\pi n^3}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{-4}{\pi n^3}\left[(-1)^n - 1\right]$$

$$\boxed{b_n = \begin{cases} \dfrac{8}{\pi n^3}, & if\ n = 1,3,5,\ldots \\ 0, & if\ n = 2,4,6,\ldots \end{cases}}$$

The required Fourier sine series be

$$(1) \Rightarrow f(x) = \sum_{n=1,3,5\ldots}^{\infty} \frac{8}{\pi n^3}\sin nx$$

$$f(x) = \frac{8}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^3}\sin nx \quad ------(2)$$

Deduction:

Let $x = \frac{\pi}{2}$ be a point of continuity.

$$(2) \Rightarrow f\left(\frac{\pi}{2}\right) = \frac{8}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi}{2}\right) \qquad \because f(x) = x(\pi - x) \Rightarrow f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2} - \frac{\pi^2}{4} = \frac{2\pi^2 - \pi^2}{4}$$

$$\Rightarrow \frac{\pi^2}{4} = \frac{8}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi}{2}\right) \qquad \Rightarrow f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$$

$$\Rightarrow \sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^3}\sin\left(\frac{n\pi}{2}\right) = \frac{\pi^3}{32}$$

$$\Rightarrow \frac{1}{1^3}\sin\frac{\pi}{2} + \frac{1}{3^3}\sin\frac{\pi}{2} + \frac{1}{5^3}\sin\frac{\pi}{2} + \ldots = \frac{\pi^3}{32}$$

$$\Rightarrow \boxed{\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \ldots = \frac{\pi^3}{32}}$$

12. Find the half – range cosine series for f (x) = $(x - 1)^2$ in (0, 1). Hence show that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}$.

Solution:

Given:

Here ℓ = Upper Limit – Lower Limit = 1-0 = 1 $\quad \therefore\ \ell = 1$

Let the cosine series be $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell}$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos n\pi x \quad \because l=1-----(1)$$

To Find a_0:

$$a_0=\frac{2}{\ell}\int_0^{\ell} f(x)\ dx=\frac{2}{1}\int_0^1\left(x^2-2x+1\right)dx=2\left[\frac{x^3}{3}-2\frac{x^2}{2}+x\right]_0^1=2\left(\frac{1}{3}-1+1\right)=\frac{2}{3}$$

$$\boxed{a_0=\frac{2}{3}}$$

To Find a_n:

$$a_n=2\int_0^1(x^2-2x+1)\cos n\pi x dx$$

$$=2\left[\cancel{(x^2-2x+1)\left(\frac{\sin n\pi x}{n\pi}\right)}-(2x-2)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)+(2)\cancel{\left(\frac{\sin n\pi x}{n^3\pi^3}\right)}\right]_0^1$$

$$=2\left[\left(\frac{1}{n^2\pi^2}\right)(2x-2)\cos n\pi x\right]_0^1$$

$$=\frac{2}{n^2\pi^2}\left[(0)-(-2\cos 0)\right]$$

$$a_n=\frac{4}{n^2\pi^2}$$

$$(1)\Rightarrow f(x)=\frac{2/3}{2}+\sum_1^{\infty}\frac{4}{n^2\pi^2}\cos n\pi x$$

$$\therefore \boxed{f(x)=\frac{1}{3}+\frac{4}{\pi^2}\sum_1^{\infty}\frac{1}{n^2}\cos n\pi x} \quad ----(2)$$

Deduction:

Let $x=0$ be a point of discontinuity

$$(2)\ \Rightarrow f(0)=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos 0$$

$$\Rightarrow \frac{1}{2}=\frac{1}{3}+\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} \qquad \because f(x)=(x-1)^2 \Rightarrow f(0)=\frac{f(0)+f(1)}{2}=\frac{1+0}{2}=\frac{1}{2}$$

$$\Rightarrow \frac{1}{2}-\frac{1}{3}=\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \frac{1}{6}\times\frac{\pi^2}{4}=\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \boxed{\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+....=\frac{\pi^2}{24}}$$

13. Find the Fourier cosine series for $x(\pi-x)$ in $0<x<\pi$. Hence show that $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3}+....=\frac{\pi^4}{90}$.

Solution:

Given:

Let $f(x)=x(\pi-x),\ \ 0<x<\pi$

$$f(x)=\pi x-x^2$$

Here ℓ = Upper Limit – Lower Limit = π-0 = π $\quad\therefore\ \ell=\pi$

Let the cosine series be $f(x)=\dfrac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\dfrac{n\pi x}{\ell}$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \quad \because l=\pi -----(1)$$

To Find a_0:

$$a_0=\frac{2}{\ell}\int_0^{\ell}f(x)\,dx=\frac{2}{\pi}\int_0^{\pi}(\pi x-x^2)\,dx=\frac{2}{\pi}\left[\frac{\pi x^2}{2}-\frac{x^3}{3}\right]_0^{\pi}=\frac{2}{\pi}\left[\left(\frac{\pi^3}{2}-\frac{\pi^3}{3}\right)-(0)\right]$$

$$=\frac{2}{\pi}\left[\frac{3\pi^3-2\pi^3}{6}\right]=\frac{2}{\pi}\left[\frac{\pi^3}{6}\right]=\frac{\pi^2}{3}$$

$$\boxed{a_0=\frac{\pi^2}{3}}$$

To Find a_n:

$$a_n=2\int_0^1(x^2-2x+1)\cos n\pi x\,dx$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}(\pi x-x^2)\cos nx\,dx$$

$$=\frac{2}{\pi}\left[\left(\pi x-x^2\right)\left(\frac{\sin nx}{n}\right)-\left(\pi-2x\right)\left(\frac{-\cos nx}{n^2}\right)+(-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\left(\frac{1}{n^2}\right)\left(\pi-2x\right)\cos nx\right]_0^{\pi}$$

$$=\frac{2}{\pi n^2}\left[-\pi\cos n\pi-\pi\cos 0\right]=\frac{-2\pi}{\pi n^2}\left[(-1)^n+1\right]$$

$$=\frac{-2}{n^2}\left[(-1)^n+1\right]$$

$$\boxed{a_n=\begin{cases}-\dfrac{4}{n^2}, & \text{if } n=2,4,6...\\ 0, & \text{if } n=1,3,5...\end{cases}}$$

$$\therefore \boxed{f\left(x\right)=\frac{\pi^2}{6}+\sum_{n=2,4,...}^{\infty}-\frac{4}{n^2}\cos nx.}$$

Deduction:

Let the Parseval's identity for Fourier cosine series be

$$\frac{2}{l}\int_0^{l}\left[f\left(x\right)\right]^2dx=\frac{{a_0}^2}{2}+\sum_{n=1}^{\infty}a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi}\left[\pi x-x^2\right]^2dx=\frac{\frac{\pi^4}{9}}{2}+\sum_{n=2,4,6,...}^{\infty}\left(\frac{-4}{n^2}\right)^2$$

$$\frac{2}{\pi}\int_0^{\pi}\left[\pi^2x^2-2\pi x^3+x^4\right]dx=\frac{\pi^4}{18}+16\sum_{n=2,4,6,...}^{\infty}\frac{1}{n^4}$$

$$\frac{2}{\pi}\left[\left(\frac{\pi^5}{3}-\frac{\pi^5}{2}+\frac{\pi^5}{5}\right)-(0)\right]=\frac{\pi^4}{18}+16\sum_{n=1}^{\infty}\frac{1}{(2n)^4} \qquad \because \sum_{n=2,4,6,..}^{\infty}\frac{1}{n}=\sum_{n=1}^{\infty}\frac{1}{2n}$$

$$\frac{2\pi^5}{\pi}\left(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\right)=\frac{\pi^4}{18}+\frac{16}{16}\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$2\pi^4\left(\frac{10-15+6}{30}\right)-\frac{\pi^4}{18}=\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\frac{\pi^4}{15}-\frac{\pi^4}{18}=\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{6\pi^4-5\pi^4}{90}$$

$$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\dots=\frac{\pi^4}{90}}$$

14. Find the cosine series for f (x) = x in (0, π) and then using Parseval's theorem, show that $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\dots.=\frac{\pi^4}{96}$.

Solution:

Given:

Let $f(x)=x,\ 0<x<\pi$

Here ℓ = Upper Limit – Lower Limit = π-0 = π $\quad\therefore\ \ell=\pi$

Let the cosine series be $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{\ell}$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\pi \qquad \because l=\pi$$

To Find a_0:

$$a_0=\frac{2}{\ell}\int_0^{\ell}f(x)\,dx=\frac{2}{\pi}\int_0^{\pi}x\,dx=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}=\frac{2}{2\pi}\left[\pi^2-0\right]=\pi$$

$$\boxed{a_0=\pi}$$

To Find a_n:

$$a_n=\frac{2}{\ell}\int_0^{\ell}f(x)\cos\frac{n\pi x}{l}\,dx=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx$$

$$=\frac{2}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right)-(1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$=\frac{2}{\pi}\left[\left(\frac{1}{n^2}\right)\cos nx\right]_0^{\pi}$$

$$=\frac{2}{\pi n^2}\left[\cos n\pi-\cos 0\right]$$

$$=\frac{2}{n^2\pi}\left[(-1)^n-1\right]$$

$$a_n = \begin{cases} \dfrac{-4}{n^2\pi}, & if \;\; n = 1,3,5,\ldots \\ 0, & if \;\; n = 2,4,6,\ldots \end{cases}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1,3,5\ldots}^{\infty} \frac{-4}{\pi n^2} \cos nx$$

Deduction:

Let the Parseval's identity for Fourier cosine series be

$$\frac{2}{l}\int_0^l \left[f(x) \right]^2 dx = \frac{a_0{}^2}{2} + \sum_{n=1}^{\infty} a_n^2$$

$$\frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{\pi^2}{2} + \sum_{n=1,3,5,\ldots}^{\infty} \left(\frac{-4}{n^2\pi} \right)^2$$

$$\frac{2}{\pi}\left(\frac{x^3}{3} \right)_0^{\pi} = \frac{\pi^2}{2} + \sum_{n=1,3,5\ldots}^{\infty} \frac{16}{\pi^2 n^4}$$

$$\frac{2}{3\pi}\left(\pi^3 - 0 \right) = \frac{\pi^2}{2} + \frac{16}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^4}$$

$$\frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{16}{\pi^2} \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^4}$$

$$\frac{\pi^2}{6} \times \frac{\pi^2}{16} = \sum_{n=1,3,5\ldots}^{\infty} \frac{1}{n^4}$$

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots\infty = \frac{\pi^4}{96}$$

15. Find the complex form of the Fourier series of $f(x) = \cos ax$ in $(-\pi, \pi)$, where a is not an integer.

Solution:

Given $f(x) = \cos ax$ **in** $(-\pi, \pi)$

$$\text{Here } \ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{\pi - (-\pi)}{2} = \frac{2\pi}{2} \qquad \therefore \; \ell = \pi$$

Let the complex form of the Fourier series be

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \; e^{\frac{in\pi x}{l}}$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \; e^{inx} \qquad \because l = \pi$$

$$C_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$$

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos ax \, e^{-inx} dx$$

$$\because \int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2}\left[a\cos bx + b\sin bx \right] \qquad \text{Here } a = -in \;\; \& \;\; b = a$$

$$= \frac{1}{2\pi}\left\{ \frac{e^{-inx}}{\left((in)^2 - a^2 \right)} \left(-in\cos ax + a\sin ax \right) \right\}_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi\left(a^2-n^2\right)}\left[e^{-in\pi}\left(-in\cos\ a\pi + a\sin a\pi\right) - e^{in\pi}\left(-in\cos a\pi - a\sin a\pi\right)\right]$$

$$\boxed{e^{in\pi} = \cos n\pi + i\sin n\pi = \cos n\pi = (-1)^n \qquad \text{and } e^{-in\pi} = \cos n\pi - i\sin n\pi = \cos n\pi = (-1)^n \quad \because \sin n\pi = 0}$$

$$= \frac{1}{2\pi\left(a^2-n^2\right)}\left[(-1)^n\left[-in\cos\ a\pi + a\sin a\pi + in\cos a\pi + a\sin a\pi\right]\right]$$

$$= \frac{1}{2\pi\left(a^2-n^2\right)}\left[(-1)^n(2a\sin a\pi)\right]$$

$$\boxed{C_n = \frac{(-1)^n a\sin a\pi}{\pi\left(a^2-n^2\right)}}$$

$$\therefore \boxed{f(x) = \frac{a\sin a\pi}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n}{\left(a^2-n^2\right)}e^{inx}}$$

16. Find complex form of the Fourier series of the function $f(x) = e^{-x}, -1 < x < 1$

Solution:

Given $f(x) = e^{-x}, -1 < x < 1$

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{1-(-1)}{2} = \frac{2}{2} = 1 \qquad \therefore\ \ell = 1$

Let the complex form of the Fourier series be

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} C_n\ e^{\frac{in\pi x}{l}}}$$

$$f(x) = \sum_{n=-\infty}^{\infty} C_n\ e^{in\pi x} \qquad \because l = 1$$

$$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-\frac{in\pi x}{l}}dx = \frac{1}{2}\int_{-1}^{1} e^{-x}e^{-in\pi x}dx$$

$$C_n = \frac{1}{2}\int_{-1}^{1} e^{-x-in\pi x}dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi x)}dx$$

$$= \frac{1}{2}\left\{\frac{e^{-(1+in\pi x)}}{-(1+in\pi)}\right\}_{-1}^{1}$$

$$= -\frac{-1}{2(1+in\ \pi)}\left\{e^{-(1+in\pi)} - e^{(1+in\pi x)}\right\}$$

$$= -\frac{-1}{2(1+in\ \pi)}\left\{e^{-1}e^{-in\pi} - e^{1}e^{in\pi}\right\}$$

$$\boxed{e^{in\pi} = \cos n\pi + i\sin n\pi = \cos n\pi = (-1)^n \qquad \text{and } e^{-in\pi} = \cos n\pi - i\sin n\pi = \cos n\pi = (-1)^n \quad \because \sin n\pi = 0}$$

$$= \frac{-(1-in\pi)}{2\left(1+n^2\ \pi^2\right)}\left\{e^{-1}(-1)^n - e^{1}(-1)^n\right\}$$

$$= \frac{(1-in\pi)}{2(1+n^2\pi^2)}(-1)^n\left(e^1 - e^{-1}\right)$$

$$\boxed{C_n = \frac{(1-in\,\pi)}{2(1+n^2\pi^2)}(-1)^n\, 2\sinh(1)} \qquad \because \sinh\theta = \frac{e^\theta - e^{-\theta}}{2}$$

$$\therefore \boxed{f(x) = \sum_{-\infty}^{\infty} \frac{(1-in\,\pi)}{(1+n^2\pi^2)}(-1)^n \sinh(1)\, e^{in\pi x}}$$

17. Calculate the first two harmonic of the Fourier series of $f(x)$ from the following data

x	0	30	60	90	120	150	180	210	240	270	300	330
f(x)	1.8	1.1	0.3	0.16	0.5	1.3	2.16	1.25	1.3	1.52	1.76	2.0

Solution:

x	0	30	60	90	120	150	180	210	240	270	300	330	~~360~~
f(x)	1.8	1.1	0.3	0.16	0.5	1.3	2.16	1.25	1.3	1.52	1.76	2.0	~~1.8~~

We know that $360 = 2\pi$

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{2\pi - 0}{2} = \pi \qquad \therefore\ \ell = \pi$

K=12

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \because \ell = \pi$$

$$\boxed{f(x) = \frac{a_0}{2} + (a_1\cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots}$$

Where

$a_0 = \frac{2}{K}\sum y \quad b_1 = \frac{2}{K}\sum y\sin x$

$a_1 = \frac{2}{K}\sum y\cos x \quad b_2 = \frac{2}{K}\sum y\sin 2x$

$a_2 = \frac{2}{K}\sum y\cos 2x$

x	y	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	1.8	1.8	1.8	0	0
30	1.1	0.95	0.55	0.55	0.95
60	0.3	0.15	-0.15	0.26	0.26
90	0.16	0.00	-0.16	0.16	0.00
120	0.5	-0.25	-0.25	0.43	-0.43
150	1.3	-1.13	0.65	0.65	-1.12
180	2.16	-2.16	2.16	0.00	0.01
210	1.25	-1.08	0.62	-0.63	1.08
240	1.3	-0.65	-0.65	-1.13	1.12

270	1.52	0.00	-1.52	-1.52	-0.01
300	1.76	0.88	-0.87	-1.52	-1.53
330	2.0	1.73	1.01	-1.00	-1.73
Total	15.15	0.26	3.18	-3.74	-1.39

$a_0 = \frac{2}{12}(15.15) = 2.52$ $b_1 = \frac{2}{12}(-3.74) = -0.62$

$a_1 = \frac{2}{12}(0.26) = 0.043$ $b_2 = \frac{2}{12}(-1.39) = -0.23$

$a_2 = \frac{2}{12}(3.18) = 0.53$

$$\boxed{f(x) = 1.26 + (0.043\cos x - 0.62\sin x) + (0.53\cos 2x - 0.23\sin 2x) + ...}$$

18. Find the first two harmonic of the Fourier series of f (x) given by

x	**0**	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
f (x)	**1**	**1.4**	**1.9**	**1.7**	**1.5**	**1.2**	**1.0**

Solution:

x	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
f (x)=y	1	1.4	1.9	1.7	1.5	1.2	1.0

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{2\pi - 0}{2} = \pi$ $\quad\therefore\ \ell = \pi$

K=6

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{\ell}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \because \ell = \pi$$

$$\boxed{f(x) = \frac{a_0}{2} + (a_1\cos x + b_1 \sin x) + (a_2\cos 2x + b_2\sin 2x) + ...}$$

Where

$a_0 = \frac{2}{K}\sum y$ $b_1 = \frac{2}{K}\sum y\sin x$

$a_1 = \frac{2}{K}\sum y\cos x$ $b_2 = \frac{2}{K}\sum y\sin 2x$

$a_2 = \frac{2}{K}\sum y\cos 2x$

x	y	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	1	1	1	0	0
$\frac{\pi}{3} = 60$	1.4	0.7	-0.7	1.212	1.212
$\frac{2\pi}{3} = 120$	1.9	-0.95	-0.95	1.65	-1.645
$\pi = 180$	1.7	-1.7	1.7	0	0
$\frac{4\pi}{3} = 240$	1.5	-0.75	-0.75	-1.299	1.299

$\frac{5\pi}{3}=300$	1.2	0.6	-0.6	-1.039	-1.039
Total	8.7	-1.1	-0.3	0.5196	-0.1732

$a_0=\frac{2}{6}(8.7)=2.9\ \ b_1=\frac{2}{6}(0.5196)=0.17$

$a_1=\frac{2}{6}(-1.1)=-0.37\ \ b_2=\frac{2}{6}(-0.1732)=-0.06$

$a_2=\frac{2}{6}(-0.3)=-0.1$

$$\boxed{f(x)=1.45+(-0.37\cos x+0.17\sin x)+(-0.1\cos 2x-0.06\sin 2x)+\ldots}$$

19. Find the first three harmonic of the Fourier series of f (x) given by

X	0	1	2	3	4	5
f (x)	9	18	24	28	26	20

Solution:

X	0	1	2	3	4	5	6
f (x)	9	18	24	28	26	20	9

Here $\ell=\frac{\text{Upper Limit}-\text{Lower Limit}}{2}=\frac{6-0}{2}=3\qquad \therefore\ \ell=3$

K=6

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{\ell}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{\ell}$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{3}+\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{3}$$

$$\boxed{f(x)=\frac{a_0}{2}+\left(a_1\cos\frac{\pi x}{3}+b_1\sin\frac{\pi x}{3}\right)+\left(a_2\cos\frac{2\pi x}{3}+b_2\sin\frac{2\pi x}{3}\right)+\left(a_3\cos\frac{3\pi x}{3}+b_3\sin\frac{3\pi x}{3}\right)+\ldots}$$

Where

$a_0=\frac{2}{K}\sum y$ $a_1=\frac{2}{K}\sum y\cos\left(\frac{\pi x}{3}\right)$ $a_2=\frac{2}{K}\sum y\cos\left(\frac{2\pi x}{3}\right)$ $a_3=\frac{2}{K}\sum y\cos\left(\frac{3\pi x}{3}\right)$	$b_1=\frac{2}{K}\sum y\sin\left(\frac{\pi x}{3}\right)$ $b_2=\frac{2}{K}\sum y\sin\left(\frac{2\pi x}{3}\right)$ $b_3=\frac{2}{K}\sum y\sin\left(\frac{3\pi x}{3}\right)$

x	y	$y\cos\frac{\pi x}{3}$ $(or)\ y\cos 60x$	$y\cos\frac{2\pi x}{3}$ $y\cos 120x$	$y\cos\frac{3\pi x}{3}$ $y\cos 180x$	$y\sin\frac{\pi x}{3}$ $y\sin 60x$	$y\sin\frac{2\pi x}{3}$ $y\sin 120x$	$y\sin\frac{3\pi x}{3}$ $y\cos 180x$
0	9	9	9	9	0	0	0
1	18	9	-9	-18	15.7	15.6	0
2	24	-12	-24	24	20.9	-20.784	0
3	28	-28	28	-28	0	0	0
4	26	-13	-13	26	-22.6	22.6	0

5	20	10	-10	-20	-17.4	-17.4	0
Total	125	-25	-19	-7	-3.4	20.8	0

$a_0 = \frac{2}{6}(125) = 41.66 \; b_1 = \frac{2}{6}(-3.4) = -1.13$

$a_1 = \frac{2}{6}(-25) = -8.33 \; b_2 = \frac{2}{6}(20.8) = 6.9$

$a_2 = \frac{2}{6}(-19) = -6.33 \; b_3 = \frac{2}{6}(0) = 0$

$a_3 = \frac{2}{6}(-7) = -2.3$

The required Harmonic of the Fourier series be

$$f(x) = \frac{41.66}{2} + \left(-8.33\cos\frac{\pi x}{3} + -1.13\sin\frac{\pi x}{3}\right) + \left(-6.33\cos\frac{2\pi x}{3} + 6.9\sin\frac{2\pi x}{3}\right) + \left(-2.3\cos\frac{3\pi x}{3} + 0\right) + \ldots$$

$$\boxed{f(x) = 20.83 + \left(-8.33\cos\frac{\pi x}{3} + -1.13\sin\frac{\pi x}{3}\right) + \left(-6.33\cos\frac{2\pi x}{3} + 6.9\sin\frac{2\pi x}{3}\right) + \left(-2.3\cos\frac{3\pi x}{3}\right) + \ldots}$$

20. The following table gives the variations of a periodic function over a period T

x	***0***	***T/6***	***T/3***	***T/2***	***2T/3***	***5T/6***	***T***
f(x)	**1.98**	**1.3**	**1.05**	**1.3**	**- 0.88**	**- 0.25**	**1.98**

find***f(x)*****upto first harmonic.**

Solution:

Assume $X = \frac{2\pi x}{T}$

X	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$	2π
y=f(θ)	**1.98**	**1.3**	**1.05**	**1.3**	**- 0.88**	**- 0.25**	**1.98**

Here $\ell = \frac{\text{Upper Limit} - \text{Lower Limit}}{2} = \frac{2\pi - 0}{2} = \pi \qquad \therefore \; \ell = \pi$

K=6

$$f(X) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi X}{\ell} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi X}{\ell}$$

$$f(X) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nX + \sum_{n=1}^{\infty} b_n \sin nX$$

$$\boxed{f(x) = \frac{a_0}{2} + (a_1\cos nX + b_1\sin nX) + (a_2\cos 2X + b_2\sin 2X) + (a_3\cos 3X + b_3\sin 3X) + \ldots}$$

Where

$a_0 = \frac{2}{K}\sum y \; a_1 = \frac{2}{K}\sum y\cos X$				$b_1 = \frac{2}{K}\sum y\sin X$
x	y	$y\cos X$	$y\sin X$	
0	1.98	1.98	0	
$\frac{\pi}{3}$	1.30	0.65	1.1258	

$\frac{2\pi}{3}$	1.05	-0.525	0.9093
π	1.30	-1.3	0
$\frac{4\pi}{3}$	-0.88	0.44	0.762
$\frac{5\pi}{3}$	-0.25	-0.125	0.2165
Total	4.6	1.12	3.013

$$a_0 = \frac{2}{6}\sum y = \frac{4.6}{3} = 1.5 \quad a_1 = \frac{2}{6}(1.12) = 0.37 \quad b_1 = \frac{2}{6}(3.013) = 1.005$$

$$\therefore \boxed{f(x) = 0.75 + 0.37\cos X + 1.005\sin X}$$