## Poisson Process

Let $X(t)$ denotes the number of occurrences of a certain event in the interval $(0, t)$. Then the discrete random process $\{X(t)\}$ is called the Poisson process if it satisfies the following postulates.

### Postulates of Poisson process

(i) $P[1\ occurrence\ in\ (t, t + \Delta t)] = \lambda \Delta t + O(\Delta t) = P_1(\Delta t)$

(ii) $P[0\ occurrence\ in\ (t, t + \Delta t)] = 1 - \lambda \Delta t + O(\Delta t) = P_0(\Delta t)$

(iii) $P[2\ or\ more\ occurrences\ in\ (t, t + \Delta t)] = O(t)$

### Some Applications of Poisson Process:

(i) Arrival of customers in simple queuing system

(ii) The number of wrong telephone calls at a switch board.

(iii) The number of passengers entering a railway station on a given day.

(iv) The emission of radioactive particles.

### Probability law of Poisson process:

Let $\lambda$ be the number of customers per unit time. $P_m(t)$ be the probability of m occurences, of the event in the interval (0, t).

$P_0(t)$ denotes the probability of 0 occurences, in the interval (0, t).

Probability of zero occurrence of the event in the time interval 0 to $t + \Delta t$ is given by $P_0(t + \Delta t)$.

Also $P_0(t+\Delta t)$ can be rewritten as

$P_0(t+\Delta t)$ = Probability of zero occurrence in the interval $(0, t)$ and also in the interval $(t, t+\Delta t\ )$

i.e., $P_0(t+\Delta t) = P_0(1-\lambda\Delta t)$

$$= P_0(t) - \lambda\Delta t P_0(t)$$

$$\Rightarrow P_0(t+\Delta t) - P_0(t) = -\lambda\Delta t P_0(t)$$

$$\Rightarrow \frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -\lambda P_0(t)$$

Taking lim as $\Delta t \to 0$

$$\Rightarrow \lim_{n\to 0} \frac{P_0(t+\Delta t)-P_0(t)}{\Delta t} = -\lambda P_0(t)$$

$$\Rightarrow \frac{d}{dt}[P_0(t)] = -\lambda P_0(t)$$

$$\Rightarrow \frac{d[P_0(t)]}{P_0(t)} = -\lambda dt$$

Integrating on both sides

$$\Rightarrow \int \frac{d[P_0(t)]}{P_0(t)} = \int -\lambda dt$$

$$\Rightarrow \log P_0(t) = -\lambda t + c$$

$$\Rightarrow P_0(t) = e^{-\lambda t} A \text{ where } A = e^c \qquad ...(1)$$

To find A, Put $t = 0$

$\Rightarrow P_0(0) = A$

$\Rightarrow A = 1$

$(1) \Rightarrow P_0(0) = e^{-\lambda t}$

Now $P_m(t) = P[X(t) = m]$

= P( $m$ occurrences of the event in $(0, t)$)

$P_m(t + \Delta t) = P[X(t + \Delta t) = m]$

= P( $m$ occurrences of the event in $(0, t + \Delta t)$)

= P(m – 1 occurrences in $(0, t)$ and 1 occurrence in $(t, t + \Delta t)$)+ P( $m$ occurrences of the event in $(0, t)$ and 0 occurrence in $(t, t + \Delta t)$)

$= P_{m-1}(t)\lambda\Delta t + P_m(t)(1 - \lambda\Delta t)$

$= P_{m-1}(t)\lambda\Delta t + P_m(t) - \lambda P_m(t)\Delta t$

$\Rightarrow P_m(t + \Delta t) - P_m(t) = \lambda\Delta t[P_{m-1}(t) - P_m(t)]$

$\Rightarrow \frac{P_m(t+\Delta t) - P_m(t)}{\Delta t} = \lambda[P_{m-1}(t) - P_m(t)]$

Taking lim as $\Delta t \to 0$

$\lim_{\Delta t \to 0} \frac{P_m(t+\Delta t) - P_m(t)}{\Delta t} = \lambda[P_{m-1}(t) - P_m(t)]$

$\Rightarrow \frac{d}{dt}[P_m(t)] = \lambda[P_{m-1}(t) - P_m(t)]$

$\Rightarrow \frac{d}{dt} P_m(t) + \lambda P_m(t) = \lambda P_{m-1}(t)$

$\Rightarrow P_m(t)e^{\lambda t} = \int \lambda P_{m-1}(t)\, e^{\lambda t} dt$

Put $m = 1$

$\Rightarrow P_1(t)e^{\lambda t} = \int \lambda P_0(t)\, e^{\lambda t} dt$

$= \int \lambda e^{-\lambda t}\, e^{\lambda t} dt$

$= \int \lambda\, dt$

$\Rightarrow P_1(t)e^{\lambda t} = \lambda dt$

$\Rightarrow P_1(t)e^{\lambda t} = \frac{(\lambda t)^1 e^{-\lambda t}}{1!}$

This is the first order Poisson probability distribution.

In general $P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$

**Properties of Poisson Process:**

**Property:1**

**Poisson Process is a Markov process.**

**Proof:**

Let us take the conditional probability distribution of $X(t_3)$ given the past values of $X(t_2)$ and $X(t_1)$ .

Assume that $t_3 > t_2 > t_1$ and $n_3 > n_2 > n_1$

Consider $P[X(t_3) = n_3/X(t_2) = n_2, X(t_1) = n_1\,]$

$$= \frac{P[X(t_3) = n_3, X(t_2) = n_2, X(t_1) = n_1]}{P[X(t_1) = n_1, X(t_2) = n_2]}$$

$$= \frac{e^{-\lambda(t_3-t_2)} . \lambda^{n_3-n_2} \cdot (t_3-t_2)^{n_3-n_2}}{(n_2-n_3)!}$$

$$= P[X(t_3) = n_3 / X(t_2) = n_2 ]$$

Hence the conditional probability distribution $X(t_3)$ given that values of the process $X(t_2)$ and $X(t_1)$ depends only on the most recent value $X(t_2)$ of the process.

Hence Poisson process is a Markov process.

Hence the proof.

**Property: 2**

**Additive Property**

**The sum of two independent Poisson processes is a Poisson Process.**

**Proof:**

Let $X_1(t)$ and $X_2(t)$ be two independent Poisson process with parameter $\lambda_1 t$ and $\lambda_2 t$ respectively.

Let $X(t) = X_1(t) + X_2(t)$

Now $P[X(t) = n] = P\big[\big(X_1(t) + X_2(t)\big) = n\big]$

$\Rightarrow \sum_{r=0}^{n} P[X_1(t) = r]P[X_2(t) = n - r]$

$\Rightarrow \sum_{r=0}^{n} \frac{e^{-\lambda_1 t}(\lambda_1 t)^r}{r!} \frac{e-^{\lambda_2 t}(\lambda_2 t)^{n-r}}{(n-r)!}$

$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} (\lambda_1 t)^r (\lambda_2 t)^{n-r}$

$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} \sum_{r=0}^{n} nC_r (\lambda_1 t)^r (\lambda_2 t)^{n-r}$

$= \frac{e^{-(\lambda_1+\lambda_2)t}}{n!} (\lambda_1 t + \lambda_2 t)^n$

Which is a Poisson process with parameter $\lambda_1 + \lambda_2$.

Hence the proof.

**Property: 3**

**Difference of two independent Poisson processes not a Poisson Process.**

**Proof:**

Let $X_1(t)$ and $X_2(t)$ be two independent Poisson process with parameter $\lambda_1 t$ and $\lambda_2 t$ respectively.

Let $X(t) = X_1(t) + X_2(t)$

Now $E[X(t)] = E[X_1(t)] - E[X_2(t)]$

$= \lambda_1 t - \lambda_2 t$

Now, $E[X^2(t)] = E\left[\left(X_1(t) - X_2(t)\right)^2\right]$

$$= E\left[{X_1}^2 t + {X_2}^2 t - 2X_1(t)X_2(t)\right]$$

$$= E\left[{X_1}^2 t\right] + E\left[{X_2}^2 t\right] - 2E[X_1(t)X_2(t)]$$

$$= {\lambda_1}^2 t^2 + \lambda_1 t + {\lambda_2}^2 t^2 + \lambda_2 t - 2E[X_1(t)]E[X_2(t)]$$

$$= {\lambda_1}^2 t^2 + {\lambda_2}^2 t^2 + \lambda_1 t + +\lambda_2 t - 2\lambda_1 t \lambda_2 t$$

$$= (\lambda_1 + \lambda_2)t + \left({\lambda_1}^2 + {\lambda_2}^2\right)t^2 - 2\lambda_1\lambda_2 t^2$$

$$= (\lambda_1 + \lambda_2)t + \left({\lambda_1}^2 + {\lambda_2}^2 - 2\lambda_1\lambda_2\right)t^2$$

$$= (\lambda_1 + \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2$$

$$\neq (\lambda_1 + \lambda_2)t + (\lambda_1 - \lambda_2)^2 t^2$$

Hence $X(t) = X_1(t) + X_2(t)$ is not a Poisson process.

Hence the proof.

**Property: 4**

**Distribution of the interval arrival time**

**The inter arrival time of a Poisson process with parameter $\lambda$ follows exponential distribution with mean $\frac{1}{\lambda}$.**

**Proof:**

Let the two consecutive events be $E_i$ and $E_{i+1}$.

Let $E_i$ take place at time $t_i$ and $E_{i+1}$ take place at time $t_i + T$.

Let T be the interval between the occurrences $E_i$ and $E_{i+1}$ then T is a continuous random variable.

Now, $P(T > t) = P(t < T)$

$= P(E_{i+1}$ does not occur in $(t_i, t_i + T\ ))$

= P(no event occurs in the interval of length t)

$= P[X(t) = 0]$

$= \frac{e^{-\lambda t}(\lambda t)^0}{0!}$

$= e^{-\lambda t}$

The cumulative distribution of T is

$F(t) = P(T \leq t) = 1 - P(T > t)$

$= 1 - e^{-\lambda t}$

Hence the probability density function is

$$f(t) = \frac{d}{dt}[F(t)]$$

$= -e^{-\lambda t}(-\lambda)$

$$= \lambda e^{-\lambda t}$$

Which is the probability density function of exponential distribution with mean $\frac{1}{\lambda}$.

Hence the proof.

**Property: 5**

**If the number of occurrences of an event E in an interval of length t is a Poisson process $X(t)$ with parameter $\lambda t$ and if such occurrences of E has a constant probability of being recorded and the recording are independent of each other. Then the number $N(t)$ of the recorded occurrences in time t is also a Poisson process.**

**Proof:**

$$P[N(t) = n] = \sum_{r=0}^{n} [E\ occurs\ (n + r) times\ and\ n\ of\ them\ are\ recorded]$$

$$= \sum_{r=0}^{n} \frac{e^{-\lambda t}(\lambda t)^r}{(n+r)!} (n + r)C_n p^n q^{n+r-n}$$

$$= e^{-\lambda t} \sum_{r=0}^{n} \frac{(\lambda t)^n (\lambda t)^r}{(n+r)!} \frac{(n+r)!}{n!(n+r-n)!} p^n q^r$$

$$= \frac{e^{-\lambda t}(\lambda t)^n}{n!} p^n \sum_{r=0}^{n} \frac{(\lambda t q)^r}{r!}$$

$$= \frac{e^{-\lambda t}(\lambda tp)^n}{n!}\left[1 + \frac{\lambda tp}{1!} + \frac{(\lambda tp)^2}{2!} + \dots\right]$$

$$= \frac{e^{-\lambda t}(\lambda tp)^n}{n!} e^{\lambda tq}$$

$$= \frac{e^{-\lambda t(1-q)}(\lambda tp)^n}{n!}$$

$$= \frac{e^{-\lambda tp}(\lambda tp)^n}{n!}$$

= Probability density function of Poisson process

Hence $N(t)$ is a Poisson process.

Hence the proof.

**Problems under Probability law of Poisson Process:**

**1. Suppose the customer arrive at a bank according to a Poisson Process with mean rate of 3 per minute. Find the probability that during a time interval of two minutes. (i) Exactly 4 customer arrive. (ii)Greater than 4 customer arrive. (iii)Fewer than 4 customers arrive.**

**Solution:**

Let $X(t)$ denotes the number of customers arrived during the interval $(0, t)$.

Then $\{X(t)\}$ follows Poisson process.

Given : $\lambda = 3/\min$ $and\ t = 2\ min.$

$P(X(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, n = 0,1,2,3 \dots\dots..$

$i.e)., P(X(2) = n) = \frac{e^{-6}(6)^n}{n!}, n = 0,1,2,3 \dots..$

(i)P(Exactly 4 customer arrive during a time interval of two minutes)

$P(X(2) = 4) = \frac{e^{-6}(6)^4}{4!} = \frac{3.2124}{24} = 0.13385$

$P(X(2) = 4) = 0.13385$

(ii)P( greater than 4 customer arrive during a time interval of two minutes) =

$P(X(2) > 4) = 1 - P(X(2) \le 4)$

$= 1 - \{P(X(2) = 0) + P(X(2) = 1) + P(X(2) = 2) +$

$P(X(2) = 3) + P(X(2) = 4)\}$

$$= 1 - \left[\frac{e^{-6}(6)^0}{0!} + \frac{e^{-6}(6)^1}{1!} + \frac{e^{-6}(6)^2}{2!} + \frac{e^{-6}(6)^3}{3!} + \frac{e^{-6}(6)^4}{4!}\right]$$

$$= 1 - e^{-6}\left[1 + 6 + \frac{36}{2} + 36 + 54\right]$$

$$= 1 - e^{-6}[115] = 1 - 0.285 = 0.715$$

$P(X(2) > 4) = 0.715$

(iii)P(less than 4 customer arrive during a time interval of two minutes)

$P(X(2) < 4) = P(X(2) = 0) + P(X(2) = 1) + P(X(2) = 2) + P(X(2) = 3)$

$$= \left[\frac{e^{-6}(6)^0}{0!} + \frac{e^{-6}(6)^1}{1!} + \frac{e^{-6}(6)^2}{2!} + \frac{e^{-6}(6)^3}{3!}\right]$$

$$= e^{-6}[1 + 6 + 8 + 36]$$

$$= e^{-6}[61] = 0.1512$$

$P(X(2) < 4) = 0.1512$

**2. A hard disk fails in a computer system it follows a Poisson distribution with mean rate of 1 per week. Find the probability, that 2 weeks have elapsed since last failure. If we have 5 extra hard disks and the next supply is not due in 10 weeks. Find the probability that the machine will not be out of order in the next 10 weeks.**

**Solution:**

Given $\lambda = 1$

$P(X(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, n = 0,1,2,3 \ldots$

To find the probability that 2 weeks have elapsed since last failure.

$P(X(2) = 0) = \frac{e^{-1\times 2}(1\times 2)^0}{0!}$

$$= e^{-2}$$

$$= 0.1353$$

Number of extra hard disk = 5

The probability that the machine will not be the probability that the machine will not be out of order in the next weeks is $P(X(10) \leq 5)$

$$= \{P(X(10) = 0) + P(X(10) = 1) + P(X(10) = 2) + P(X(10) = 3) + P(X(10) = 4) + P(X(10) = 5)\}$$

$$= \frac{e^{-10}(10)^0}{0!} + \frac{e^{-10}(10)^1}{1!} + \frac{e^{-10}(10)^2}{2!} + \frac{e^{-10}(10)^3}{3!} + \frac{e^{-10}(10)^4}{4!} + \frac{e^{-10}(10)^5}{5!}$$

$$= e^{-10}\left[1 + 10 + \frac{10^2}{2!} + \frac{10^3}{3!} + \frac{10^4}{4!} + \frac{10^5}{5!}\right]$$

$$= 0.067$$

**3. A fisher man catches fish independently at a Poisson rate of 2 from a large lake with a lot fish. If he starts at $10.00$am, what is the hour pribability that he catches 1 fish by $10.30$am and 3 fishes by noon?**

**Solution:**

Let $X(t)$ denotes the number of fishes caught by the fisherman in $(0, t)$. Then $\{X(t)\}$ follows a Poisson process. Given $\lambda = 2$ per hr.

$$P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}; n = 0,1,2 \ldots \ldots \ldots \infty$$

$$= \frac{e^{-2t}(2t)^n}{n!}; n = 0,1,2 \dots \dots \dots \infty \; \lambda = 2$$

Given a fisherman starts catching at 10.00am and fisherman catches fish independently.

$\therefore P$[ he catches one fish in 10.30am and 3 fishes in noon]

$= P$[ he catches one fish in 10.30am]$P$[ he catches 3 fishes in noon]

$= P$ [he catches one fish in 30mins]$P$[ he catches 3 fishes at 12pm]

$= P$ [he catches one fish in 30mins]$P$[ he catches 3 fishes in 2 hours ]

$= P$[ he catches one fish by $\frac{1}{2}$ an hour ]$P$[ he catches 3 fishes in 2 hours]

[$\because$ $t$ is in hours]

$$= P\left[X\left(\frac{1}{2}\right) = 1\right] P[X(2) = 3] = \frac{e^{-2\frac{1}{2}}\left(2\left(\frac{1}{2}\right)\right)^1}{1!} \frac{e^{-4}4^3}{3!} = e^{-1}\frac{32e^{-4}}{3}$$

$P$[ he catches 3 fishes in noon ] $= 0.07188$

**4. On the average, a submarine on patrol sights 6 enemy ship per hour. Assume that the number of ships sighted in a given length of time is a Poisson variate, find the probability of sighting (1) 6 ships in the next half an hour (2) 4 ships in the next 2 hours (3) at least one ship in the next 15 minutes.**

**Solution:**

Let $X(t)$ denote the number of submarine on patrol sights in the interval $(0, t)$.

Then $\{X(t)\}$ follows Poisson process.

Given $\lambda = 6$ per hour

$$P[X(t) = n] = \frac{e^{-\lambda t}(\lambda t)^n}{n!}; n = 0,1,2, \dots .,\infty$$

$$= \frac{e^{-6t}(6t)^n}{n!}; n = 0,1,2, \dots ,\infty$$

(i) $P[$ sighting 6 ships next half an hour $] = P\left[X\left(\frac{1}{2}\right) = 6\right]$

$$= \frac{e^{-6\left(\frac{1}{2}\right)}\left(6\left(\frac{1}{2}\right)\right)^6}{6!} = \frac{e^{-3}3^6}{6!} = 0.0504$$

(ii) $P[$ sighting 4 ships in the next 2 hours $] = P[X(2) = 4]$

$$= \frac{e^{-6(2)}(6(2))^4}{n!} = \frac{e^{-12}12^4}{4!} = 0.0053$$

(iii) $P[$ sighting at least one ship in the next $15$mins $] = P[$ at least one ship in the next $\frac{1}{4}hr]$

$$= P\left[X\left(\frac{1}{4}\right) \geq 1\right] = 1 - P\left[X\left(\frac{1}{4}\right) < 1\right] = 1 - P\left[X\left(\frac{1}{4}\right) = 0\right]$$

$$= 1 - \frac{e^{-6\left(\frac{1}{4}\right)}\left(6\left(\frac{1}{4}\right)\right)^0}{0!}$$

$$= 1 - e^{-\frac{3}{2}} = 0.975$$

**Problems under $P(N(t) = n) = \frac{e^{-\lambda pt}(\lambda pt)^n}{n!}$**

**1. A radioactive source emits particles at rate of five per minute in accordance with Poisson process. Each particle emitted has a probability 0.6 of being recorded. Find the probability that 10 particles are recorded in a 4 minutes period.**

**Solution:**

Given: $\lambda = 5/\min \ and \ t = 4 , n = 10$

Probability of recording : $p = 0.6$

Let $N(t)$ be the number of particles recorded during the interval $(0, t)$

Then $\{N(t)\}$ follows a Poisson process with parameter $\lambda p$.

$$P(N(t) = n) = \frac{e^{-\lambda pt}(\lambda pt)^n}{n!}$$

$$= \frac{e^{-3t}(3t)^n}{n!} \ ; n = 0,1,2,3, ....$$

P( 10 particles are recorded in a 4 minute period) = $P(N(4) = 10) = \frac{e^{-12}(12)^{10}}{10!}$

$P(N(4) = 10) = 0.1048$

**2.If customer arrives at a counter in accordance with a Poisson process with a mean rate of 2 per minute, find the probability that the interval between two**

**consecutive arrivals is (i) more than 1 minute (ii) between 1 minute and 2 minutes and (iii)4 minutes or less.**

**Solution:**

Given: $\lambda = 2/\min$

Let $T$ be the interval between two consecutive arrivals.

The exponential distribution is $f(t) = 2e^{-2t}, t > 0$

(i)$P[T > 1] = \int_1^\infty f(t)\, dt$

$P[T > 1] = \int_1^\infty 2e^{-2t}\, dt = 2\left[\frac{e^{-2t}}{-2}\right]_1^\infty = -[e^{-\infty} - e^{-2}]$

$P[T > 1] = e^{-2} = 0.1353$

(ii) $P[1 < T < 2] = \int_1^2 f(t)\, dt$

$$= \int_1^2 2e^{-2t}\, dt = 2\left[\frac{e^{-2t}}{-2}\right]_1^2 = -[e^{-4} - e^{-2}]$$

$$= -[0.0183 - 0.1353]$$

$$= 0.117$$

(iii)$P[T \leq 4] = \int_0^4 f(t)\, dt$

$$= \int_0^4 2e^{-2t}\, dt = 2\left[\frac{e^{-2t}}{-2}\right]_0^4 = -[e^{-8} - e^{-0}]$$

$$= -[3.35 * 10^{-4} - 1]$$

$$= 0.996$$

$$P[T \leq 4] = 0.996$$