2.5 Generating Function:

The generating function for the sequence "s" with terms $a_0, a_1, \ldots, a_n$ of real numbers is the infinite sum.

$$G(x) = G(s,x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots$$

$$= \sum_{n=0}^{\infty} a_n x^n$$

For example, (i) The generating function for the sequence "s" with the terms $1, 1, 1, \ \ . \ \ . \ \ .$ is given by

$$G(x) = G(s,x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

(ii) The generating function for the sequence "s" with terms $1, 2, 3, 4, \ \ . \ \ . \ \ .$ is given by

$$G(x) = G(s,x) = \sum_{n=0}^{\infty} (n+1)x^n$$

$$= 1 + 2x + 3x^2 + \ldots$$

$$= (1-x)^{-2}$$

$$= \frac{1}{(1-x)^2}$$

Problems:

1. Write the generating function for the sequence $1, a, a^2, a^3, a^4, \ldots$

Solution:

Generating function $G(x) = 1 + a + a^2 + a^3 + a^4 + \ldots$

$$= \frac{1}{1-ax} \text{ for } |ax| < 1$$

Solution for Recurrence Relations using Generating Functions:

Procedure for solving Recurrence Relation using Generating Function:

Step: 1 Rewrite the recurrence relation as an equation on RHS

Step: 2 Multiply the equation in step: 1 by x^n and summing it from 1 to ∞ or (0 to ∞) or (2 to ∞)

Step: 3 Put $G(x) = \sum_{n=0}^{\infty} a_n x^n$ and write $G(x)$ as a function of x.

Step: 4 Decompose $G(x)$ into partial fraction.

Step: 5 Express $G(x)$ as a sum of familiar series.

Step: 6 Express a_n as the coefficient of x^n in $G(x)$.

The following table represents some sequences and their generating functions.

S. no	Sequence	Generating Function
1	1	$\frac{1}{1-z}$
2	$(-1)^n$	$\frac{1}{1+z}$
3	a^n	$\frac{1}{1-az}$
4	$(-a)^n$	$\frac{1}{1+az}$
5	$n+1$	$\frac{1}{1-(z)^2}$
6	n	$\frac{1}{(1-z)^2}$
7	n^2	$\frac{z(1+2)}{(1-z)^3}$
8	na^n	$\frac{az}{(1-za)^2}$

1. Using generating function solve the recurrence relation $a_n = 3a_{n-1}$ for $n \geq 1$ with $a_0 = 2$

Solution:

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$

Given $a_n - 3a_{n-1} = 0$

Multiply the above equation by x^n and summing from 1 to ∞, we get

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - \sum_{n=1}^{\infty} 3a_{n-1}x^n = 0$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n x^n - 3x \sum_{n=1}^{\infty} 3a_{n-1}x^{n-1} = 0$$

$\Rightarrow (G(x) - a_0) - 3xG(x) = 0$

$\Rightarrow G(x)(1 - 3x) = a_0$

$\Rightarrow G(x)(1 - 3x) = 2$

$\Rightarrow G(x) = \frac{2}{(1-3x)} = 2(1 - 3x)^{-1}$

$= 2(1 + 3x + (3x)^2 + \ \dots)$

$= 2\sum_{n=0}^{\infty} 3^n x^n$

Consequently, $a_n = 2$. coefficient of x^n in $G(x)$

$a_n = 2 \cdot 3^n$

2. Solve the recurrence relation $a_n - 7a_{n-1} + 10a_{n-2} = 0$ for $n \geq 2$ given that $a_0 = 10, a_1 = 41$ using generating function.

Solution:

The given recurrence relation is $a_n - 7a_{n-1} + 10a_{n-2} = 0$

Multiply the above equation by x^n and summing from 2 to ∞, we get

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 7\sum_{n=2}^{\infty} a_{n-1}x^n + 10\sum_{n=2}^{\infty} a_{n-2}x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - 7x\sum_{n=2}^{\infty} a_{n-1}x^{n-1} + 10x^2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 0$$

$\Rightarrow [G(x) - a_0 - a_1 x] - 7x[G(x) - a_0] + 10x^2 G(x) = 0$

$\Rightarrow G(x) - 10 - 41x - 7x[G(x) - 10] + 10x^2 G(x) = 0$

$\Rightarrow G(x)(1 - 7x + 10x^2) + 29x - 10 = 0$

$\Rightarrow G(x) = \frac{10-29x}{10x^2-7x+1}$

$\Rightarrow G(x) = \frac{10-29x}{(1-2x)(1-5x)}$

$\Rightarrow G(x) = \frac{A}{(1-2x)} + \frac{B}{(1-5x)}$

$= A(1-2x)^{-1} + B(1-5x)^{-1}$

$= A[1 + 2x + (2x)^2 + \dots] + B[1 + 5x + (5x)^2 + \dots]$

$= A\sum_{n=2}^{\infty} 2^n x^n + B\sum_{n=2}^{\infty} 5^n x^n$

a_n = coefficient of x^n in $G(x)$

$a_n = A2^n + B5^n, n \geq 2 \quad \ldots (A)$

Given $a_0 = 10$, Put $n = 0$ in (A), we get

$\Rightarrow a_0 = A2^0 + B5^0$

$\Rightarrow 10 = A + B \quad \ldots (1)$

Given $a_1 = 41$, Put $n = 1$ in (A), we get

$\Rightarrow a_1 = A2^1 + B5^1$

$\Rightarrow 41 = 2A + 5B \quad \ldots (2)$

Solving (1) and (2) we get $A = 3, B = 7$

Hence $a_n = 3 \cdot 2^n + 7 \cdot 5^n$

3. Using generating function solve the recurrence relation corresponding to the Fibonacci sequence $a_n = a_{n-1} + a_{n-2}, n \geq 2$ with $a_0 = 1, a_1 = 1$

Solution:

Given recurrence relation $a_n - a_{n-1} - a_{n-2} = 0$

Multiply the above recurrence relation by x^n and summing from 2 to ∞, we get

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - \sum_{n=2}^{\infty} a_{n-1} x^n - \sum_{n=2}^{\infty} a_{n-2} x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n x^n - x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} - x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0$$

$$\Rightarrow [G(x) - a_0 - a_1 x] - x[G(x) - a_0] - x^2 G(x) = 0$$

$$\Rightarrow G(x) - 10 - 41x - 7x[G(x) - 10] + 10x^2 G(x) = 0$$

$$\Rightarrow G(x)(1 - x - x^2) = a_0 - a_0 x + a_1 x$$

$$\Rightarrow G(x) = \frac{1}{1-x-x^2}$$

$$= \frac{1}{\left(1-\frac{1+\sqrt{5}}{2}x\right)\left(1-\frac{1-\sqrt{5}}{2}x\right)}$$

$$= \frac{A}{\left(1-\frac{1+\sqrt{5}}{2}x\right)} + \frac{B}{\left(1-\frac{1-\sqrt{5}}{2}x\right)}$$

Now $\frac{1}{1-x-x^2} = \frac{A}{\left(1-\frac{1+\sqrt{5}}{2}x\right)} + \frac{B}{\left(1-\frac{1-\sqrt{5}}{2}x\right)}$ . . . (1)

$1 = A\left(1 - \frac{1-\sqrt{5}}{2}x\right) + B\left(1 - \frac{1+\sqrt{5}}{2}x\right)$ . . . (2)

Put $x = 0$ in (2)

(2) $\Rightarrow A + B = 1$ . . . (3)

Put $x = \frac{2}{1-\sqrt{5}}$ in (2)

(2) $\Rightarrow 1 = B\left(1 - \frac{1+\sqrt{5}}{1-\sqrt{5}}\right)$

$$\Rightarrow 1 = B\left(\frac{1-\sqrt{5}-1-\sqrt{5}}{1-\sqrt{5}}\right)$$

$$\Rightarrow 1 = B\left(\frac{-2\sqrt{5}}{1-\sqrt{5}}\right) \quad \text{(Using B value in (3))}$$

$$\Rightarrow B = \frac{1-\sqrt{5}}{-2\sqrt{5}}$$

$$(3) \Rightarrow A = \frac{1}{2\sqrt{5}}\left(1+\sqrt{5}\right)$$

Sub A and B in (1), we get

$$G(x) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)\left(1-\left(\frac{1+\sqrt{5}}{2}\right)x\right)^{-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)\left(1-\left(\frac{1-\sqrt{5}}{2}\right)x\right)^{-1}$$

$$= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)\left[1+\frac{1+\sqrt{5}}{2}x+\left(\frac{1+\sqrt{5}}{2}\right)^2+\dots\right]$$

$$-\frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)\left[1+\frac{1-\sqrt{5}}{2}x+\left\{\left(\frac{1-\sqrt{5}}{2}\right)x\right\}^2+\dots\right]$$

a_n = coefficient of x^n in $G(x)$

Solving, we get

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

4. Identify the sequence having the expression $\frac{5+2x}{1-4x^2}$ as a generating function.

Solution:

Given $G(x) = \frac{5+2x}{1-4x^2}$ . . . (1)

$$= \frac{5+2x}{(1+2x)(1-2x)}$$

Now, $\frac{5+2x}{(1+2x)(1-2x)} = \frac{A}{(1+2x)} + \frac{B}{(1-2x)}$ . . . (2)

Put $x = \frac{1}{2}$

$\Rightarrow 5 + 1 = 2B$

$\Rightarrow B = 3$

Put $x = -\frac{1}{2}$

$\Rightarrow 5 - 1 = 2A$

$\Rightarrow A = 2$

(2) $\Rightarrow \frac{5+2x}{(1+2x)(1-2x)} = \frac{2}{(1+2x)} + \frac{3}{(1-2x)}$

$$= 2(1-2x)^{-1} + 3(1-2x)^{-1}$$

$$= A[1 - 2x - (2x)^2 + \ldots] + B[1 + 2x + (2x)^2 + \ldots]$$

$$= 2\sum_{n=2}^{\infty}(-1)^n 2^n x^n + 3\sum_{n=2}^{\infty} 2^n x^n$$

$$= 2\sum_{n=2}^{\infty}(-2)^n x^n + 3\sum_{n=2}^{\infty} 2^n x^n$$

The required sequence is the coefficient of x^n in $G(x)$

Hence $S(n) = 2(-2)^n + 3(2)^n$

5. Identify the sequence having the expression $\frac{3-5x}{1-2x-3x^2}$ as a generating function.

Solution:

Given $G(x) = \frac{3-5x}{1-2x-3x^2}$ . . . (1)

$= \frac{3-5x}{(1-3x)(1+x)}$

Now, $\frac{3-5x}{(1+2x)(1-2x)} = \frac{A}{(1-3x)} + \frac{B}{(1+x)}$ . . . (2)

$3 - 5x = A(1 + x) + B(1 - 3x)$

Put $x = -1$

$\Rightarrow 3 + 5 = 4B$

$\Rightarrow B = 2$

Put $x = \frac{1}{3}$

$\Rightarrow 3 - \frac{5}{3} = A\left(1 + \frac{1}{3}\right)$

$\Rightarrow \frac{4}{3} = \frac{4}{3}A$

$\Rightarrow A = 1$

$(2) \Rightarrow \frac{3-5x}{(1+2x)(1-2x)} = \frac{1}{(1-3x)} + \frac{2}{(1+x)}$

$$= (1-3x)^{-1} + 2(1+x)^{-1}$$

$$= A[1 + 3x + (3x)^2 + \ldots] + B[1 - x + (x)^2 + \ldots]$$

$$= \sum_{n=2}^{\infty} 3^n x^n + 3\sum_{n=2}^{\infty}(-1)^n x^n$$

The required sequence is the coefficient of x^n in $G(x)$

Hence $S(n) = 3^n + 2(-1)^n$