

UNIT-II

FOURIER SERIES

PROBLEMS BASED ON ODD AND EVEN FUNCTIONS

Even and Odd functions

A function $f(x)$ is said to be even if $f(-x) = f(x)$. For example x^2 , $\cos x$, $x \sin x$, $\sec x$ are even functions. A function $f(x)$ is said to be odd if $f(-x) = -f(x)$. For example, x^3 , $\sin x$, $x \cos x$, etc., are odd functions.

(1) The Euler's formula for even function is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where $a_0 = \frac{2}{\pi} \int_0^\pi f(x) dx$; $a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx$

The Euler's formula for odd function is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx$

Example 1

Find the Fourier Series for $f(x) = x$ in $(-\pi, \pi)$

Here, $f(x) = x$ is an odd function.

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{---(1)}$$

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx \\ &= \frac{2}{\pi} \int_0^\pi x d \left[\frac{-\cos nx}{n} \right] \\ &= \frac{2}{\pi} \left[x \left(\frac{-\cos nx}{n} \right) - (1) \left(\frac{-\sin nx}{n^2} \right) \right]_0^\pi \end{aligned}$$

$$= \frac{2}{\pi} \left[\frac{-\pi \cos n\pi}{n} \right] \quad \left\{ \begin{array}{l} \\ \\ \end{array} \right.$$

$$b_n = \frac{2(-1)^{n+1}}{n}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

$$\text{i.e., } x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx$$

Example 7

Expand $f(x) = |x|$ in $(-\pi, \pi)$ as FS and hence deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

Solution

Here $f(x) = |x|$ is an even function. a_0

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad (1)$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi} x \frac{\sin nx}{n} dx \right]$$

$$= \frac{2}{\pi} \left\{ \left(x \frac{\sin nx}{n} \right)_0^{\pi} - \left(1 \frac{-\cos nx}{n^2} \right)_0^{\pi} \right\}$$

$$= \frac{2}{\pi} \left\{ \frac{\cos n\pi - 1}{n^2} \right\}$$

$$a_n = \frac{2}{\pi n^2} [(-1)^n - 1]$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2} [(-1)^n - 1] \cos nx$$

Putting $x = 0$ in equation (2), we get

$$0 = \frac{\pi}{2} + \frac{4}{\pi} + \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \quad \left. \right\}$$

$$\text{Hence, } \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

Example 8

$$\text{If } f(x) = 1 + \frac{2x}{\pi} \text{ in } (-\pi, 0)$$

$$= 1 - \frac{2x}{\pi} \text{ in } (0, \pi)$$

Then find the FS for $f(x)$ and hence show that $\sum_{n=1}^{\infty} (2n-1)^{-2} = \pi^2/8$

Here $f(-x)$ in $(-\pi, 0) = f(x)$ in $(0, \pi)$

$$f(-x) \text{ in } (0, \pi) = f(x) \text{ in } (-\pi, 0)$$

$\therefore f(x)$ is an even function

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{--- (1).}$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \frac{2x}{\pi} dx \quad \left. \right\}$$

$$= \frac{2}{\pi} \left[x - \frac{2x^2}{2\pi} \right]_0^{\pi} \quad \left. \right\}$$

$$a_0 = 0$$

$$\begin{aligned}
 a_n &= \frac{2}{\pi} \int_0^\pi \frac{2x}{1 - \frac{2x}{\pi}} \cos nx dx \\
 &= \frac{2}{\pi} \left[\left(1 - \frac{2x}{\pi} \right) \frac{\sin nx}{n} \right]_0^\pi - \left[\frac{-2}{\pi} \right] \left(\frac{-\cos nx}{n^2} \right)_0^\pi \\
 a_n &= \frac{4}{\pi^2 n^2} [(1 - (-1)^n] \\
 \therefore f(x) &= \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2} [1 - (-1)^n] \cos nx \\
 &= \frac{4}{\pi^2} \left(\frac{2 \cos x}{1^2} + \frac{2 \cos 3x}{3^2} + \frac{2 \cos 5x}{5^2} \right) + \dots \quad (2)
 \end{aligned}$$

Put $x = 0$ in equation (2) we get

$$\begin{aligned}
 \frac{\pi^2}{4} &= 2 + \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \\
 \Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots &= \frac{\pi^2}{8} \\
 \text{or } \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} &= \frac{\pi^2}{8}
 \end{aligned}$$

Example 9

Obtain the FS expansion of $f(x) = x \sin x$ in $(-\pi < x < \pi)$ and hence deduce that

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots = \frac{\pi - 2}{4}.$$

Here $f(x) = x \sin x$ is an even function.

$$\text{Let } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad (1)$$

$$\text{Now, } a_0 = \frac{2}{\pi} \int_0^{\pi} x \sin x \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \, d(-\cos x)$$

$$= \frac{2}{\pi} (x)(-\cos x) - (1)(-\sin x) \Big|_0^{\pi}$$

$$a_0 = 2$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \left[\sin(1+n)x + \sin(1-n)x \right] dx$$

$$= \frac{1}{\pi} \int_0^{\pi} x \, d \left\{ \frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n} \right\}$$

$$= \frac{1}{\pi} \left\{ (x) \left\{ \frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n} \right\} - (1) \left\{ \frac{-\sin(1+n)x}{(1+n)^2} - \frac{\sin(1-n)x}{(1-n)^2} \right\} \right\} \Big|_0^{\pi}$$

$$= \frac{1}{\pi} \left\{ \frac{-\pi \cos(1+n)\pi}{1+n} - \frac{\pi \cos(1-n)\pi}{1-n} \right\}$$

$$- [\cos \pi \cos n\pi - \sin \pi \sin n\pi] \quad [\cos \pi \cos n\pi - \sin \pi \sin n\pi]$$

$$= \frac{(1+n)(-1)^n + (1-n)(-1)^n}{1-n^2}$$

$$a_n = \frac{2(-1)^n}{-n^2}, \text{ Provided } n \neq 1$$

When $n = 1$

$$a_1 = \frac{2}{\pi} \int_0^\pi x \sin x \cos x dx$$

$$= \frac{1}{\pi} \int_0^\pi x \sin 2x dx$$

$$\begin{aligned} &= \frac{1}{\pi} \int_0^\pi x d \left[\frac{-\cos 2x}{2} \right] \\ &= \frac{1}{\pi} \left(x \left[\frac{-\cos 2x}{2} \right] - (1) \left[\frac{-\sin 2x}{4} \right] \right) \Big|_0^\pi \end{aligned}$$

Therefore, $a_1 = -1/2$

$$f(x) = a_0 + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{1}{2} \cos x + \sum_{n=2}^{\infty} \frac{2(-1)^n}{1-n^2} \cos nx$$

$$\text{ie, } x \sin x = \frac{1}{2} \cos x - 2 \frac{\cos 2x}{3} - \frac{\cos 3x}{8} + \frac{\cos 4x}{15} - \dots$$

Putting $x = \pi/2$ in the above equation, we get

$$\frac{\pi}{2} = 2 \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} - \dots \quad \left. \right\}$$

$$\text{Hence, } \frac{1}{1.3} + \frac{1}{1.5} + \frac{1}{5.7} = \frac{\pi-2}{4}$$

Exercises:

Determine Fourier expressions of the following functions in the given interval:

i. $f(x) = \pi/2 + x, -\pi \leq x \leq 0$

$\pi/2 - x, 0 \leq x \leq \pi$

ii. $f(x) = -x+1$ for $-\pi \leq x \leq 0$
 $x+1$ for $0 \leq x \leq \pi$

iii. $f(x) = |\sin x|, -\pi \leq x \leq \pi$

iv. $f(x) = x^3$ in $-\pi \leq x \leq \pi$

v. $f(x) = x \cos x, -\pi < x < \pi$

vi. $f(x) = |\cos x|, -\pi < x < \pi$

vii. Show that for $-\pi < x < \pi$, $\sin ax = \frac{2\sin a\pi}{\pi} - \frac{\sin x}{1^2 - a^2} + \frac{2\sin 2x}{2^2 - a^2} - \frac{3\sin 3x}{3^2 - a^2} \dots$