BA4201 / QUANTITATIVE TECHNIQUES FOR DECISION MAKING

UNIT – III : GAME THEORY :: TWO PERSON ZERO SUM GAME

Introduction

A game with only two-persons is said to be two-person zero-sum game if the gain of one player is equalto the loss of the other so that total sum is zero. In a two-person game, the payoffs in terms of gains or losses, when players selecttheir particular strategies can be represented in the form of a matrix, called the payoff matrix of the player.If the game is zero-sum, the gain of one player is equal to the loss of the other and vice-versa. So, one player's payoff table would contain the same amounts I payoff table of the other payer with the sign changed. If the player A has strategies A_1, A_2, …, A_n and the player B has strategies B_1, B_2, …, B_n and if a_{ij} represent the payoffs that the player A gains from player B when player A chooses strategy i, and player B chooses strategy j then payoff matrix for player A is given by

Player B's strategies

$$\text{Player } A\text{'s strategies} \quad \begin{array}{c} \\ A_1 \\ A_2 \\ \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} B_1 & B_1 & \ldots\ldots & B_n \end{array} \\ \begin{bmatrix} a_{11} & a_1 & \ldots\ldots & a_{1n} \\ a_{21} & a_{21} & \ldots\ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix} \end{array}$$

Pure Strategies: Games with Saddle Point

For payer A, the minimum value in each row represents the least gain to him if he chooses his particular strategy. He will then select the strategy that gives the largest gain among the row minimum values. Thischoice of player A is called the **maximin principle** and the corresponding gain is called the maximin valueof the game denoted by $\underline{v}$.

For player B, who is assumed to be loser, the maximum value in each column represents the maximum loss to value in each column represents the maximum loss to him if he chooses his particular strategy. Hewill then select the strategy that gives minimum loss among the column maximum values. This choice ofplayer B is called the **minimax principle** and the corresponding loss is called the minimax value of the game, denoted by $\bar{v}$.

Saddle point. A saddle point of a payoff matrix is that position in the payoff matrix where maximum of row minima coincides with the minimum of the column maxima. The saddle point need not be unique.

Value of the game. The amount of payoff at the saddle point is called the value of the game, denoted by $\underline{v}$.

Fair game. A game is said to be fair if $\underline{v} = 0 = \bar{v}$.

Strictly determinable game. A game is said to be strictly determinable if $\underline{v} = v = \bar{v}$.

Procedure to Determine Saddle Point

a) Select the minimum element in each row and enclose it in a rectangle box.

b) Select the maximum element in each column and enclose it in a circle.

c) Find the element which is enclosed by the rectangle as well as the circle such element is thevalue of the game and that position is a saddle point.

Problem : 1 For the game with payoff matrix:

		Player B		
		B_1	B_2	B_3
Player A	A_1	-1	2	-2
	A_2	6	4	-6

Determine the optimal strategies for players A and B.Also determine the value of game. Is this game (i) fair? (ii) strictly determinable?

Solution. Select the row minimum and enclose it in a rectangle. Then select the column maximum and enclose it in a circle.

$$
\begin{array}{cccc}
 & B_1 & B_2 & B_3 \\
A_1 & -1 & 2 & \textcircled{\boxed{-2}} \\
A_2 & \textcircled{6} & 4 & \boxed{-6}
\end{array}
$$

Saddle point is (A_1, B_3) .Value of game = –2

Optimal strategy for A is A_1 and for B is B_3.

The game is strictly determinable. Since value of game is not zero, the game is not fair.

Problem 2:.

$$
\text{Player } A \quad
\begin{array}{c}
 \\
A_1 \\ A_2 \\ A_3
\end{array}
\overset{\text{Player } B}{\overset{\begin{array}{ccc} B_1 & B_2 & B_3 \end{array}}{\begin{bmatrix} 1 & 3 & 1 \\ 0 & -4 & -3 \\ 1 & 5 & -1 \end{bmatrix}}}
$$

Solve the game hide payoff matrix is given by

Solution. Select the row minimum and enclose it in a rectangle select the column maximum and encloseit in a circle.

$$
\text{Player } A \quad
\begin{array}{c}
 \\
A_1 \\ A_2 \\ A_3
\end{array}
\overset{\text{Player } B}{\overset{\begin{array}{ccc} B_1 & B_2 & B_3 \end{array}}{\begin{bmatrix} \textcircled{\boxed{1}} & 3 & \textcircled{\boxed{1}} \\ 0 & \boxed{-4} & -3 \\ 1 & \textcircled{5} & \boxed{-1} \end{bmatrix}}}
$$

We observe that there exist two saddle points at positions (1, 1) and (1, 3). Thus, the solution of the gameis given by

(i) the optimum strategy for player A is A_1.

(ii) the optimum strategies for player B are B_1 and B_3.

(iii) the value of game is

1 for A and B.Since $v \neq 0$,

the game is not fair.

Problem 3

Consider the game G with the following payoff matrix:

$$\begin{array}{cc} & \text{Player } B \\ \text{Player } A & \begin{bmatrix} 2 & 6 \\ -2 & \lambda \end{bmatrix} \end{array}$$

(a) Show that G is strictly determinable, whatever λ may be

(b) Determine the value of G.

Solution:

$$\begin{array}{cccccc} & & \multicolumn{2}{c}{\text{Player } B} & \\ & & B_1 & B_2 & \text{Row minima} \\ \text{Player } A & \begin{matrix} A_1 \\ A_2 \end{matrix} & \multicolumn{2}{c}{\begin{bmatrix} 2 & 6 \\ -2 & \lambda \end{bmatrix}} & \begin{matrix} 2 \\ -2 \end{matrix} \\ \text{Column maxima} & & 2 & 6 & \end{array}$$

First, ignoring the value of λ , we determine the maximin and minimax values of the payoffmatrix, as shown below:

Since maximin value = 2 = minimax value, the game G is strictly determinable, whatever λ may be valueof game G is 2

Problem 4

For what value of λ , the game with following payoff matrix is strictly determinable?

	B_1	B_2	B_3
A_1	λ	6	2
A_2	−1	λ	−7
A_3	−2	4	λ

Solution.

Ignoring the value of , we determine the maximin and minimax values of the payoff matrix,as shown below

		B_1	B_2	B_3	Row minimum	
	A_1	–5	6	2	2	← Maximin
Player A	A_2	–1	6	–7	–7	
	A_3	–2	4	1	–2	
Column maximum		–1	6	2		
		↑ Minimax				

Here maximin value = 2, minimax value = –1.

The value of game lies between –1 and 2.

For strictly determinable game, since maximin value equals minimax value, 1   2.

Problem 5:

Solve the game with the following pay-off matrix.

Player B Strategies

		I	*II*	*III*	*IV*	*V*
	1	2	5	3	6	7
Player A Strategies	2	4	6	8	1	6
	3	8	2	3	5	4
	4	15	14	18	12	20

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	-3
2	-1
3	2
4	12

Maximum of {-3, -1, 2, 12} = 12

Next consider the maximum of each column.

Column	Maximum Value
1	15
2	14
3	18
4	12
5	20

Minimum of {15, 14, 18, 12, 20}= 12

We see that the maximum of row minima = the minimum of the column maxima. So thegame has a saddle point. The common value is 12. Therefore the value V of the game = 12.

Problem 6:

Solve the game with the following pay-off matrix

Player Y Strategies

Player X Strategies

	I	*II*	*III*	*IV*	*V*
	9	12	7	14	26
	25	35	20	28	30
	7	6	–8	3	2
	8	11	13	–2	1

Solution:

First consider the minimum of each row.

Row	Minimum Value
1	7
2	20
3	-8
4	-2

Maximum of {7, 20, –8, -2} = 20

Next consider the maximum of each column.

Column	Maximum Value
1	25
2	35
3	20
4	28
5	30

Minimum of {25, 35, 20, 28, 30}= 20

It is observed that the maximum of row minima and the minimum of the column maxima are equal. Hence the given the game has a saddle point. The common value is 20. This indicates that the value V of the game is 20.