UNIT III

RANDOM PROCESS

Random Variables:

A random variable, usually written X, is a variable whose possible values are numerical outcomes of a random phenomenon. Random variable consists of two types they are discrete and continuous type variable this defines discrete- or continuous-time random processes. Sample function values may take on discrete or continuous a value is defines discrete- or continuous Sample function values may take on discrete or continuous values. This defines discrete- or continuous-parameter random process.

Random Processes Vs. Random Variables:

- For a random variable, the outcome of a random experiment is mapped onto variable, e.g., a number. For a random processes, the outcome of a random experiment is mapped onto a waveform that is a function of time.Suppose that we observe a random process X(t) at some time t1 to generate the servation X(t1) and that the number of possible waveforms is finite. If Xi(t1) is observed with probability Pi, the collection of numbers {Xi(t1)}, i =1, 2, . . . , n forms a random variable, denoted by X(t1), having the probability distribution Pi, i = 1, 2, . . . , n. E[.] = ensemble average operator.

Discrete Random Variables:

A discrete random variable is one which may take on only a countable number of distinct values such as 0,1,2,3,4,... Discrete random variables are usually (but not necessarily) counts. If a random variable can take only a finite number of distinct values, then it must be discrete. Examples of discrete random variables include the number of children in a family, the Friday night attendance at a cinema, the number of patients in a doctor's surgery, the number of defective light bulbs in a box of ten.

Probability Distribution:

The probability distribution of a discrete random variable is a list of probabilities associated with each of its possible values. It is also sometimes called the probability function or the probability mass function. Suppose a random variable X may take k different values, with the probability that $X = x_i$ defined to be $P(X = x_i) = p_i$. The probabilities p_i must satisfy the following:

1: $0 < p_i < 1$ for each i

2: $p_1 + p_2 + ... + p_k = 1$.

All random variables (discrete and continuous) have a cumulative distribution function. It is a function giving the probability that the random variable X is less than or equal to x, for every value x. For a discrete random variable, the cumulative distribution function is found by summing up the probabilities.

Distribution function of the random variable X or cumulative distribution of the random variable X

Definition :

The distribution function of a random variable X defined in $(-\infty, \infty)$ is given by $F(x) = P(X \leq x) = P\{s : X(s) \leq x\}$

Note

Let the random variable X takes values $x_1, x_2, \ldots, x_n$ with probabilities $P_1, P_2, \ldots, P_n$ and let $x_1 < x_2 < \ldots < x_n$

Then we have

$F(x) = P(X < x_1) = 0, -\infty < x < x,$

$F(x) = P(X < x_1) = 0, P(X < x_1) + P(X = x_1) \quad = 0 + p_1 = p_1$

$F(x) = P(X < x_2) = 0, P(X < x_1) + P(X = x_1) + P(X = x_2) = p_1 + p_2$

$F(x) = P(X < x_n) = P(X < x_1) + P(X = x_1) + \ldots + P(X = x_n)$

$= p_1 + p_2 + \ldots\ldots + p_n \quad = 1$

2.2 Properties Of Distribution Functions

Property : 1 $P(a < X \leq b) = F(b) - F(a)$, where $F(x) = P(X \leq x)$

Property : 2 $P(a \leq X \leq b) = P(X = a) + F(b) - F(a)$

Property : 3 $P(a < X < b) = P(a < X \leq b) - P(X = b)$

$= F(b) – F(a) – P(X = b)$ by prob (1)

2.3 Probability Mass Function (Or) Probability Function

Let X be a one dimenstional discrete R.V. which takes the values x1, x2, …… To each possible outcome 'xi' we can associate a number pi.

i.e., P(X = xi) = P(xi) = pi called the probability of xi. The number pi = P(xi) satisfies the following conditions.

(i) $p(x_i) \geq 0, \forall_i$ (ii) $\sum_{i=1}^{\infty} p(x_i) = 1$

The function p(x) satisfying the above two conditions is called the probability mass function (or) probability distribution of the R.V.X. The probability distribution {xi, pi} can be displayed in the form of table as shown below.

$X = x_i$	x_1	x_2	…….	x_i
$P(X = x_i) = p_i$	p_1	p_2	…….	p_i

Notation

Let 'S' be a sample space. The set of all outcomes 'S' in S such that X(S) = x is denoted by writing X = x.

$P(X = x) = P\{S : X(s) = x\}$

|||ly $P(x \leq a) = P\{S : X() \in (-\infty, a)\}$

and $P(a < x \leq b) = P\{s : X(s) \in (a, b)\}$

$P(X = a \text{ or } X = b) = P\{(X = a) \cup (X = b)\}$

$P(X = a \text{ and } X = b) = P\{(X = a) \cap (X = b)\}$ and so on.

Theorem :1 If X1 and X2 are random variable and K is a constant then KX1, X1 + X2, X1X2, K1X1 + K2X2, X1-X2 are also random variables.

Theorem :2

If ‘X’ is a random variable and f(•) is a continuous function, then f(X) is a random variable.

Note

If F(x) is the distribution function of one dimensional random variable then

I. $0 \leq F(x) \leq 1$

II. If $x < y$, then $F(x) \leq F(y)$

III. $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

IV. $F(\infty) = \lim_{x \to \infty} F(x) = 1$

V. If ‘X’ is a discrete R.V. taking values x_1, x_2, x_3
Where $x_1 < x_2 < x_{i-1}\ x_i$ .... then
$P(X = x_i) = F(x_i) - F(x_{i-1})$

Continuous Random Variable

Definition : A R.V.’X’ which takes all possible values in a given internal is called a continuous random variable.

Example : Age, height, weight are continuous R.V.’s.

3.1 Probability Density Function

Consider a continuous R.V. ‘X’ specified on a certain interval (a, b) (which can also be a infinite interval (-∞, ∞)).

If there is a function y = f(x) such that

$$\lim_{\Delta x \to 0} \frac{P(x < X < x + \Delta x)}{\Delta x} = f(x)$$

Then this function f(x) is termed as the probability density function (or) simply density function of the R.V. 'X'.

It is also called the frequency function, distribution density or the probability density function.

The curve y = f(x) is called the probability curve of the distribution curve.

Remark

If f(x) is p.d.f of the R.V.X then the probability that a value of the R.V. X will fall in some interval (a, b) is equal to the definite integral of the function f(x) a to b.

$$P(a < x < b) = \int_a^b f(x)dx \quad \text{(or)}$$

$$P(a \le X \le b) = \int_a^b f(x)dx$$

3.2 Properties OF P.D.F

The p.d.f f(x) of a R.V.X has the following properties

1. In the case of discrete R.V. the probability at a point say at x = c is not zero. But in the case of a continuous R.V.X the probability at a point is always zero.

$$P(X = c) = \int_{-\infty}^{\infty} f(x)dx = [x]_c^C = C - C = 0$$

2. If x is a continuous R.V. then we have p(a ≤ X ≤ b) = p(a ≤ X < b)

= p(a < X V b)

IMPORTANT DEFINITIONS INTERMS OF P.D.F

If f(x) is the p.d.f of a random variable 'X' which is defined in the interval (a, b) then

i	Arithmetic mean	$\int_a^b x\, f(x)dx$
ii	Harmonic mean	$\int_a^b \frac{1}{x} f(x)dx$
iii	Geometric mean 'G' log G	$\int_a^b \log x\, f(x)dx$
iv	Moments about origin	$\int_a^b x^r f(x)dx$
v	Moments about any point A	$\int_a^b (x-A)^r f(x)dx$
vi	Moment about mean μ_r	$\int_a^b (x-\text{mean})^r f(x)dx$
vii	Variance μ_2	$\int_a^b (x-\text{mean})^2 f(x)dx$
viii	Mean deviation about the mean is M.D.	$\int_a^b \lvert x-\text{mean}\rvert f(x)dx$

Continuous Distribution Function

Definition :

If f(x) is a p.d.f. of a continuous random variable 'X', then the function

$$F_X(x) = F(x) = P(X \le x) = \int_{-\infty}^{\infty} f(x)dx, \ -\infty < x < \infty$$

is called the distribution function or cumulative distribution function of the random variable.

Properties Of CDF Of A R.V. 'X'

(i) $0 \le F(x) \le 1, -\infty < x < \infty$

(ii) $\underset{x\to-\infty}{\text{Lt}} F(x) = 0, \quad \underset{x\to-\infty}{\text{Lt}} F(x) = 1$

(iii) $P(a \le X \le b) = \int_a^b f(x)dx = F(b) - F(a)$

(iv) $F'(x) = \frac{dF(x)}{dx} = f(x) \ge 0$

(v) $P(X = x_i) = F(x_i) - F(x_i - 1)$

Moment Generating Function

Definition : The moment generating function (MGF) of a random variable 'X' (about origin) whose probability function f(x) is given by

$$M_X(t) = E[e^{tX}]$$

$$= \begin{cases} \int_{x=-\infty}^{\infty} e^{tx} f(x)dx, \text{ for a continuous probably function} \\ \sum_{x=-\infty}^{\infty} e^{tx} p(x), \text{ for a discrete probably function} \end{cases}$$

Where t is real parameter and the integration or summation being extended to the entire range of x.

Discrete Distributions

The important discrete distribution of a random variable 'X' are

1. Binomial Distribution
2. Poisson Distribution
3. Geometric Distribution

6.1 Binomial Distribution

Definition:

A random variable X is said to follow binomial distribution if its probability law is given by

P(x) = p(X = x successes) = nCx px qn-x Where x = 0, 1, 2, ……., n, p+q = 1

Note

Assumptions in Binomial distribution

i) There are only two possible outcomes for each trail (success or failure).

ii) The probability of a success is the same for each trail.

iii) There are 'n' trails, where 'n' is a constant.

iv) The 'n' trails are independent.

Passion Distribution

Definition :

A random variable X is said to follow if its probability law is given by

$$P(X = x) \qquad = p(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \qquad x = 0, 1, 2, \ldots, \infty$$

Poisson distribution is a limiting case of binomial distribution under the following conditions or assumptions.

1. The number of trails ‘n’ should e infinitely large i.e. $n \to \infty$.
2. The probability of successes ‘p’ for each trail is infinitely small.
3. $np = \lambda$, should be finite where λ is a constant.

Central Limit Theorem:

In probability theory, the central limit theorem (CLT) states that, given certain conditions, the arithmetic mean of a sufficiently large number of iterates of independent random variables, each with a well-defined expected value and well-defined variance, will be approximately normally distributed.

The Central Limit Theorem describes the characteristics of the "population of the means" which has been created from the means of an infinite number of random population samples of size (N), all of them drawn from a given "parent population". The Central Limit Theorem predicts that regardless of the distribution of the parent population:

[1] The mean of the population of means is always equal to the mean of the parent population from which the population samples were drawn.

[2] The standard deviation of the population of means is always equal to the standard deviation of the parent population divided by the square root of the sample size (N).

[3] The distribution of means will increasingly approximate a normal distribution as the size N of samples increases.

A consequence of Central Limit Theorem is that if we average measurements of a particular quantity, the distribution of our average tends toward a normal one. In addition, if a measured variable is actually a combination of several other uncorrelated variables, all of them "contaminated" with a random error of any distribution, our measurements tend to be contaminated with a random error that is normally distributed as the number of these variables increases.Thus, the Central Limit Theorem explains the ubiquity of the famous bell-shaped "Normal distribution" (or "Gaussian distribution") in the measurements domain.