

PROBLEMS UNDER BASIS

Let V be a vector space with $\dim(V) = n$. Then any basis of V contains n elements.

Let β be a set with cardinality(number of elements) $|\beta|$.

- If $|\beta| < n$ or $|\beta| > n$, then S does not form a basis of V .
- If β is a linearly independent set in V with $|\beta| = n$, then β forms a basis in V .

Example. Determine whether $(1,1,1), (1,0,1)$ forms a basis of R^3

Sol: Since $\dim(R^3) = 3$, any basis of R^3 contains three elements. Let $\beta = \{(1,1,1), (1,0,1)\}$. Since β contains two elements, β does not form a basis of R^3 .

Example 80. Show that the sets of vectors

$\{(1,2,1), (3,1,5), (-1,0,1), (1, -1,2)\}$ do not form a basis for $V_3(R)$.

Sol: Since $\dim(V_3(R)) = 3$, any basis of $V_3(R)$ contains three elements.

Let $\beta = \{(1,2,1), (3,1,5), (-1,0,1), (1, -1,2)\}$. Since β contains four elements, does not form a basis of $V_3(R)$.

Example Verify the vectors $(1, -1,2), (1, -2,1), (1,1,4)$ in R^3 forms a basis of R^3 .

Sol: Let $\beta = \{(1, -1,2), (1, -2,1), (1,1,4)\}$

$\dim(R^3) = 3$, which is finite.

In R^3 , any independent set with three elements is a basis of R^3 .

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & 4 \end{vmatrix}$$

$$= 1(-8 - 1) + 1(4 - 1) + 2(1 + 2) = 0$$

$\therefore \beta$ is a linearly dependent set in R^3 .

$\therefore \beta$ does not form a basis of R^3 .

Example. Verify the vectors $(1,2,0)$, $(2,3,0)$, $(8,13,0)$ of R^3 is a basis of R^3

Sol: Let $\beta = \{(1,2,0), (2,3,0), (8,13,0)\}$

$\dim(R^3) = 3$, which is finite.

In R^3 , any independent set with three elements is a basis of R^3 .

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 8 & 13 & 0 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ 8 & 13 & 0 \end{vmatrix} = 0$$

$\therefore \beta$ is a linearly dependent set in R^3 .

$\therefore \beta$ is not a basis of R^3

Example Verify the vectors $(2,1,0)$, $(-3,-3,1)$, $(-2,1,-1)$ in R^3 basis of R^3

Sol: Let $\beta = \{(2,1,0), (-3,-3,1), (-2,1,-1)\}$.

$\dim(R^3) = 3$, which is finite.

In R^3 , any independent set with three elements is a basis of R^3 .

$$\text{Let } A = \begin{bmatrix} 2 & 1 & 0 \\ -3 & -3 & 1 \\ -2 & 1 & -1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & 1 & 0 \\ -3 & -3 & 1 \\ -2 & 1 & -1 \end{vmatrix} = -1 \neq 0$$

$\therefore \beta$ is a linearly independent set in R^3 .

$\therefore \beta$ is a basis of R^3 .

Example. Check whether the following are basis for the space R^3

(a) $\{(1,1,-1), (2,3,4), (4,1,-1), (0,1,-1)\}$

(b) $\{(1,1,-1), (0,3,4), (0,0,-1)\}$

(c) $\{(1,2,0), (0,1,-1)\}$

Sol:

$\dim(R^3) = 3$, which is finite.

In R^3 , any independent set with three elements is a basis for R^3 .

(a) $\beta = \{(1,1,-1), (2,3,4), (4,1,-1), (0,1,-1)\}$

Since β contains four elements, it is not a basis for R^3 .

(b) $\beta = \{(1,1,-1), (0,3,4), (0,0,-1)\}$

The set contains three elements

Let $v_1 = (1,1,-1), v_2 = (0,3,4), v_3 = (0,0,-1)$

To prove S is a basis we have to prove S is a linearly independent.

Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{vmatrix} = -3 \neq 0$

$\therefore \beta$ is linearly independent in R^3

$\Rightarrow \beta$ is a basis in R^3

(c) $\beta = \{(1,2,0), (0,1,-1)\}$

Since the set contains two elements, it does not form a basis in R^3 .

Example 85. Determine $\{1 + 2x + x^2, 3 + x^2, x + x^2\}$ is a basis for $P_2(R)$. Sol:

$\dim P_2(R) = 3$, which is finite. In $P_2(R)$, any independent set with three elements is a basis.

Given $v_1 = 1 + 2x + x^2, v_2 = 3 + x^2, v_3 = x + x^2$

The vector equation is

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\alpha_1(1 + 2x + x^2) + \alpha_2(3 + x^2) + \alpha_3(x + x^2) = 0$$

$$(\alpha_1 + 3\alpha_2) + (2\alpha_1 + \alpha_3)x + (\alpha_1 + \alpha_2 + \alpha_3)x^2 = 0$$

Equating the like terms, we get

$$\alpha_1 + 3\alpha_2 = 0$$

$$2\alpha_1 + \alpha_3 = 0$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 0$$

Let A be the coefficients matrix,

$$\therefore A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -4 \neq 0$$

the system of homogenous equations have only the trivial solution

$$\alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$$

$\therefore v_1, v_2, v_3$ are linearly independent

Hence v_1, v_2, v_3 is a basis of $P_2(R)$

Therefore $\{1 + 2x + x^2, 3 + x^2, x + x^2\}$ is a basis over R .

Example 86. Let $V = P_2(R)$ and $\beta = \{1, 1 + x, 1 + x + x^2\}$. Check whether S forms a basis in V .

Sol: $\dim P_2(R) = 3$, which is finite.

In $P_2(R)$, any independent set with three elements is a basis.

Given $v_1 = 1, v_2 = 1 + x, v_3 = 1 + x + x^2$

The vector equation is

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\alpha_1(1) + \alpha_2(1 + x) + \alpha_3(1 + x + x^2) = 0$$

$$\alpha_3 + \alpha_1 + \alpha_2 + \alpha_2 x + \alpha_3 x + \alpha_3 x^2 = 0x^2 + 0x + 0$$

$$(\alpha_3 + \alpha_1 + \alpha_2) + (\alpha_2 + \alpha_3)x + \alpha_3 x^2 = 0x^2 + 0x + 0$$

Equating the like terms, we get

$$\alpha_3 + \alpha_1 + \alpha_2 = 0 \dots (1)$$

$$\alpha_2 + \alpha_3 = 0 \dots (2)$$

$$\alpha_3 = 0$$

$$(2) \Rightarrow \alpha_2 = 0$$

$$(1) \Rightarrow \alpha_1 = 0$$

$\therefore \beta$ is linearly independent set in $P_2(R)$,

Therefore β is a basis in $P_2(R)$,

Example 87. If the vectors $\{u, v, w\}$ form a basis for R^3 , show that the vectors $\{u, u - w, u + v - 2w\}$ also forms a basis for R^3 .

Sol: $\dim(R^3) = 3$, which is finite.

In R^3 , any independent set with three elements is a basis for R^3 .

Let $\beta = \{u, v, w\}$ and $\beta_1 = \{u - w, u + v - 2w\}$

Given β forms a basis for R^3 .

$\therefore \beta_1$ is a linearly independent set in R^3 .

In a finite dimensional vector space, any two bases has same number of elements.

Also in a finite dimensional vector space, any independent set with number elements $\dim(V)$ is a basis.

To prove β_1 is a basis for R^3 , it is enough to prove β_1 is a linearly independent set. The vector equation is

$$\alpha_1 u + \alpha_2(u - w) + \alpha_3(u + v - 2w) = 0$$

$$\alpha_1 u + \alpha_2 u - \alpha_2 w + \alpha_3 u + \alpha_3 v - 2\alpha_3 w = 0$$

$$(\alpha_1 + \alpha_2 + \alpha_3)u + \alpha_3 v + (-\alpha_2 - 2\alpha_3)w = 0$$

Since u, v and w are linearly independent,

$$\alpha_1 + \alpha_2 + \alpha_3 = 0 \dots\dots\dots (1)$$

$$\alpha_3 = 0$$

$$\alpha_2 - 2\alpha_3 = 0 \dots\dots\dots (2)$$

$$(2) \Rightarrow -\alpha_2 - 2(0) = 0$$

$$\alpha_2 = 0$$

$$(1) \Rightarrow \alpha_1 = 0$$

$$\therefore \alpha_1 u + \alpha_2(u - w) + \alpha_3(u + v - 2w) = 0 \Rightarrow \alpha_1 = 0, \alpha_2 = 0, \alpha_3 = 0$$

$\therefore \beta_1$ is a linearly independent set.

Hence β_1 is a basis of R^3 .

$$= \begin{bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \end{bmatrix}$$

Equating the like terms, we get

$$\alpha_1 = 2$$

$$\alpha_2 = 3$$

$$\alpha_3 = 4$$

$$\alpha_4 = -7$$

The coordinate of A relative to the usual basis is $(2,3,4, -7)$.

1.6.2. PROBLEMS UNDER BASIS AND DIMENSION OF A SUBSPACE

Let W be a subspace of a vector space V over F . To find the basis dimension of :

- From W , find linear span of W . Let it be β .
- Check β is linearly independent or not.
- If β is linearly independent set, then β forms a basis in W .
- $\dim(W) = |\beta|$

Example 91. Find the dimension of the subspace W of the vector space R^3 over R if $W = \{(a, 0,0)/a \in R\}$

Sol: Let $v \in W$. Then

$$v = (a, 0,0) = a(1,0,0)$$

$$\therefore \beta = \{(1,0,0)\} \text{ spans } \underline{W}.$$

Any set with one element is linearly independent

$$\therefore B \text{ is a linearly independent set in } W.$$

$$\therefore B = \{(1,0,0)\} \text{ is a basis of } W.$$

$$\therefore \dim(W) = 1$$

Example 92. Find the dimension of the subspace W of the vector space R^3 over

R , if $W = \{(a_1, a_2, a_3) / (2a_1 - 7a_2 + a_3 = 0)\}$

Sol: $W = \{(a_1, a_2, a_3) / (2a_1 - 7a_2 + a_3 = 0)\}$

Given $2a_1 - 7a_2 + a_3 = 0$

$$\Rightarrow a_3 = -2a_1 + 7a_2$$

Let $v \in W$. Then

$$v = (a_1, a_2, a_3)$$

$$(a_1, a_2, a_3) = a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1)$$

$$= a_1(1,0,0) + a_2(0,1,0) + (-2a_1 + 7a_2)(0,0,1)$$

$$= a_1(1,0,0) + a_2(0,1,0) - 2a_1(0,0,1) + 7a_2(0,0,1)$$

$$= (a_1, 0, 0) + (0, a_2, 0) + (0, 0, -2a_1) + (0, 0, 7a_2)$$

$$= (a_1, 0, -2a_1) + (0, a_2, 7a_2)$$

$$= a_1(1, 0, -2) + a_2(0, 1, 7)$$

$\therefore \beta = \{(1, 0, -2), (0, 1, 7)\}$ spans W i.e., $L(\beta) = W$

Next we prove that B is a linearly independent set in W .

Consider the vector equation

$$a_1v_1 + a_2v_2 = 0$$

$$a_1(1, 0, -2) + a_2(0, 1, 7) = 0$$

$$(a_1, a_2, -2a_1 + 7a_2) = 0$$

$$\Rightarrow a_1 = a_2 = 0$$

$\therefore \beta$ is a linearly independent set in W .

$\therefore \beta = \{(1,0,-2), (0,1,7)\}$ is a basis of W

Since the basis contains two elements, $\dim(W) = 2$

Example 93. Find the dimension of the subspace W of the vector space F^5 over

F , if $W = \{(a_1, a_2, a_3, a_4, a_5) / a_1 - a_3 + a_4 = 0\}$

Sol: $W = \{(a_1, a_2, a_3, a_4, a_5) / a_1 - a_3 + a_4 = 0\}$

Given $a_1 - a_3 + a_4 = 0$

$$\Rightarrow a_4 = a_3 - a_1$$

Let $v \in W$. Then

$$v = (a_1, a_2, a_3, a_4, a_5)$$

$$(a_1, a_2, a_3, a_4, a_5)$$

$$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + a_3(0,0,1,0,0) + a_4(0,0,0,1,0)$$

$$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + a_3(0,0,1,0,0) + (a_3 - a_1)(0,0,0,1,0) + a_5(0,0,0,0,1)$$

$$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + \overline{a_3}(0,0,1,0,0) + a_3(0,0,0,1,0) - a_1(0,0,0,1,0)$$

$$+ a_5(0,0,0,0,1)$$

$$= a_1(1,0,0,-1,0) + a_2(0,1,0,0,0) + a_3(0,0,1,1,0) + a_5(0,0,0,0,1)$$

$\therefore \beta = a_1(1,0,0,-1,0), (0,0,-1,0), (0,1,0,0,0), (0,0,1,1,0), (0,0,0,0,1)\}$ spans W

$$\text{i.e., } L(\beta) = W$$

Next we prove that β is a linearly independent set in W .

Consider the vector equation

$$a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = 0$$

$$a_1(1,0,0,-1,0) + a_2(0,1,0,0,0) + a_3(0,0,1,1,0) + a_4(0,0,0,0,1) = 0$$

$$(a_1, a_2, a_3, -a_1 + a_3, a_4) = 0$$

$$\Rightarrow a_1 = a_2 = a_3 = a_4 = 0$$

$\therefore \beta$ is a linearly independent set in W .

$\therefore \beta = \{(1,0,0,-1,0), (0,1,0,0,0), (0,0,1,1,0), (0,0,0,0,1)\}$ is a basis of W .

Since the basis contains four elements, $\dim(W) = 4$.

Example 94. Find the dimension of the subspace W of the vector space F^5

over R , if $W = \{(a_1, a_2, a_3, a_4, a_5) / a_2 = a_3 = a_4, a_1 + a_5 = 0\}$

Sol: $W = \{(a_1, a_2, a_3, a_4, a_5) / a_2 = a_3 = a_4 = 0, a_1 + a_5 = 0\}$

Given $a_1 + a_5 = 0$

$$\Rightarrow a_5 = -a_1$$

Also given $a_2 = a_3 = a_4$

$\therefore a_3 = a_2$ and $a_4 = a_2$

Let $v \in W$. Then

$$v = (a_1, a_2, a_3, a_4, a_5)$$

$$(a_1, a_2, a_3, a_4, a_5)$$

$$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + a_3(0,0,1,0,0) + a_3(0,0,0,1,0) + a_5(0,0,0,0,1)$$

$$= a_1(1,0,0,0,0) + a_2(0,1,0,0,0) + a_2(0,0,1,0,0) + a_2(0,0,0,1,0) \\ - a_1(0,0,0,0,1) = a_1(1,0,0,0,-1) + a_2(0,1,1,1,0)$$

$\beta = \{(1,0,0,0,-1), (0,1,1,1,0)\}$ spans W

i.e., $L(\beta) = W$

Next we prove that β is a linearly independent set in W .

Consider the vector equation

$$a_1v_1 + a_2v_2 = 0$$

$$a_1(1,0,0,0,-1) + a_2(0,1,1,1,0) = 0$$

$$(a_1, a_2, a_2, a_2, -a_1) = 0$$

$$\Rightarrow a_1 = a_2 = 0$$

$\therefore \beta$ is a linearly independent set in W .

$\therefore \beta = \{(1,0,0,0,-1), (0,1,1,1,0)\}$ is a basis of W . Since the basis contains two elements, $\dim(W) = 2$

Example 95. Find the dimension of the subspace W of the vector space R^3 over R , if $W = \{(a, b, c) : 2a + 3b = c; 7c + 9b = a\}$

Sol:

$$W = \{(a, b, c) : 2a + 3b = c; 7c + 9b = a\}$$

Given

$$2a + 3b = c$$

$$2a + 3b - c = 0$$

Also given

$$7c + 9b = a$$

$$a - 9b - 7c = 0 \dots (2)$$

Solve (1) and (2)

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -9 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0$$

$$\text{Let } A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -9 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & -21 & -12 \end{bmatrix} R_2 \rightarrow R_2 - R_1$$

$$\begin{vmatrix} 2 & 3 \\ 0 & -21 \end{vmatrix} = -42 \neq 0$$

$$R(A) = 2 < \text{the number of unknowns} = 3$$

Therefore the system has an infinite number of solutions.

From the last row, we get

$$-21b - 12c = 0$$

$$-21b = 12c$$

$$b = -\frac{4}{7}c$$

$$\text{Let } c = k$$

$$\therefore b = -\frac{4}{7}k$$

From the first equation, we get

$$2a + 3b - c = 0$$

$$2a - \frac{12}{7}k - k = 0$$

$$2a = \frac{19}{7}k$$

$$a = \frac{19}{14}k$$

where k is a parameter

$$W = \left\{ \left(\frac{19}{14}k, -\frac{4}{7}k, k \right) : k \in R \right\}$$

$$= \left\{ \left(\frac{19}{14}, -\frac{4}{7}, 1 \right) k : k \in R \right\}$$

$$\therefore \beta = \left\{ \left(\frac{19}{14}, -\frac{4}{7}, 1 \right) \right\} \text{ spans } W.$$

$$\text{i.e., } L(\beta) = W$$

Any set with one non vector is linearly independent

$$\therefore \beta \text{ is a linearly independent set in } W. \therefore \beta = \left\{ \left(\frac{19}{14}, -\frac{4}{7}, 1 \right) \right\} \text{ is a basis of } W.$$

Since the basis contains one element, $\dim(W) = 1$

Example(96) Find the dimension of the subspace W of the vector space

$$M_{2 \times 2}(R) \text{ over } R, \text{ if } W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b + c + d = 0 \right\}$$

$$\text{Sol: } W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + b + c + d = 0 \right\}$$

Given

$$a + b + c + d = 0$$

$$d = -a - b - c \dots (1)$$

Let $v \in W$. Then

$$v = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{aligned}
 \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (-a - b - c) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - a \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\
 &= a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}
 \end{aligned}$$

$$\therefore \beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\} \text{ spans } W.$$

i.e., $L(\beta) = W$

Next we prove that β is a linearly independent set in W .

Consider the vector equation

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$$

$$a_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + a_2 \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + a_3 \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = 0$$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & -a_1 - a_2 - a_3 \end{bmatrix} = 0$$

$$\Rightarrow a_1 = a_2 = a_3 = 0$$

$\therefore \beta$ is a linearly independent set in W .

$\therefore \beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$ is a basis of W .

Since the basis contains three elements, $\dim(W) = 3$

