5.3 Finding Inverse Z-transform by Convolution theorem

1. **Find inverse Z-transform of $\frac{z^2}{(z-a)^2}$ by using convolution theorem.**

Solution:

Given $Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]=?$ $\quad f(n)*g(n)=\sum^{n} f(k)g(n-k)$

By convolution theorem

$$Z^{-1}\left[F(z)\cdot G(z)\right]=Z^{-1}\left[F(z)\right]*Z^{-1}\left[G(z)\right]$$

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]=Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-a}\right]$$

$$=Z^{-1}\left[\frac{z}{z-a}\right]*Z^{-1}\left[\frac{z}{z-a}\right]$$

$$=a^n*a^n$$

$$=\sum_{k=0}^{n}a^k a^{n-k} \qquad \because f(n)*g(n)=\sum_{k=0}^{n}f(k)g(n-k)$$

$$=\sum_{k=0}^{n}\cancel{a^k}\,\cancel{a^{-k}}\,a^n$$

$$=a^n\sum_{k=0}^{n}1$$

$$Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]=a^n(n+1)\cdot 1=(n+1)a^n$$

$$\boxed{Z^{-1}\left[\frac{z^2}{(z-a)^2}\right]=(n+1)a^n}$$

2. **By using convolution theorem, show that the inverse Z-transform of $\frac{z^2}{(z+a)(z+b)}$ is**

$\frac{(-1)^n}{b-a}\left[b^{n+1}-a^{n+1}\right]$

Solution:

Given $Z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right]=?$

By convolution theorem

$$Z^{-1}\left[F(z)\cdot G(z)\right]=Z^{-1}\left[F(z)\right]*Z^{-1}\left[G(z)\right]$$

$$Z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right]=Z^{-1}\left[\frac{z}{z+a}\cdot\frac{z}{z+b}\right]$$

$$=Z^{-1}\left[\frac{z}{z+a}\right]*Z^{-1}\left[\frac{z}{z+b}\right]$$

$$=(-a)^n*(-b)^n$$

$$=\sum_{k=0}^{n}(-a)^k(-b)^{n-k} \qquad \because f(n)*g(n)=\sum_{k=0}^{n}f(k)g(n-k)$$

	$= (-1)^n \sum_{k=0}^{n} a^k b^{-k} b^n$ $= (-1)^n b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k$ $= (-1)^n b^n \left[1+\left(\frac{a}{b}\right)+\left(\frac{a}{b}\right)^2+\left(\frac{a}{b}\right)^3+\ldots+\left(\frac{a}{b}\right)^n\right]$ $= (-1)^n b^n \left[\frac{\left(\frac{a}{b}\right)^{n+1}-1}{\frac{a}{b}-1}\right] = b^n \left[\frac{\frac{a^{n+1}}{b^{n+1}}-1}{\frac{a}{b}-1}\right] = b^n \left[\frac{\frac{a^{n+1}-b^{n+1}}{b^{n+1}}}{\frac{a-b}{b}}\right]$ $= (-1)^n b^n \left[\frac{a^{n+1}-b^{n+1}}{b^{n+1}} \times \frac{b}{a-b}\right] = (-1)^n \not{b^n} \left[\frac{a^{n+1}-b^{n+1}}{\not{b^n}\,\not{b}} \times \frac{\not{b}}{a-b}\right]$ $= (-1)^n \left[\frac{a^{n+1}-b^{n+1}}{a-b}\right]$ $\boxed{Z^{-1}\left[\frac{z^2}{(z+a)(z+b)}\right] = \frac{(-1)^n}{b-a}\left[b^{n+1}-a^{n+1}\right]}$
3.	**Find** $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$ **using convolution theorem.** Solution: Given $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = ?$ By convolution theorem $Z^{-1}\left[F(z)\cdot G(z)\right] = Z^{-1}\left[F(z)\right] * Z^{-1}\left[G(z)\right]$ $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-b}\right]$ $= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-b}\right]$ $= (a)^n * (b)^n$ $= \sum_{k=0}^{n} (a)^k (b)^{n-k} \qquad \because f(n)*g(n) = \sum_{k=0}^{n} f(k)g(n-k)$ $= \sum_{k=0}^{n} a^k b^{-k} b^n$ $= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k$ $= b^n \left[1+\left(\frac{a}{b}\right)+\left(\frac{a}{b}\right)^2+\left(\frac{a}{b}\right)^3+\ldots+\left(\frac{a}{b}\right)^n\right]$

	$$=b^n\left[\frac{\left(\frac{a}{b}\right)^{n+1}-1}{\frac{a}{b}-1}\right]=b^n\left[\frac{\frac{a^{n+1}}{b^{n+1}}-1}{\frac{a}{b}-1}\right]=b^n\left[\frac{\frac{a^{n+1}-b^{n+1}}{b^{n+1}}}{\frac{a-b}{b}}\right]$$ $$=b^n\left[\frac{a^{n+1}-b^{n+1}}{b^{n+1}}\times\frac{b}{a-b}\right]=(-1)^n\,\cancel{b^n}\left[\frac{a^{n+1}-b^{n+1}}{\cancel{b^n}\,\cancel{b}}\times\frac{\cancel{b}}{a-b}\right]$$ $$=\frac{a^{n+1}-b^{n+1}}{a-b}$$ $$\boxed{Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]=\frac{a^{n+1}-b^{n+1}}{a-b}}$$
4.	**Using convolution theorem, find** $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]$ Solution: Given $Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]=?$ By convolution theorem $Z^{-1}[F(z)\cdot G(z)]=Z^{-1}[F(z)]*Z^{-1}[G(z)]$ $$Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]=Z^{-1}\left[\frac{8z^2}{2\left(z-\frac{1}{2}\right)4\left(z+\frac{1}{4}\right)}\right]=Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)}\cdot\frac{z}{\left(z-\frac{1}{4}\right)}\right]$$ $$=Z^{-1}\left[\frac{z}{\left(z-\frac{1}{2}\right)}\right]*Z^{-1}\left[\frac{z}{\left(z-\frac{1}{4}\right)}\right]$$ $$=\left(\frac{1}{2}\right)^n*\left(\frac{1}{4}\right)^n$$ $$=\sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(\frac{1}{4}\right)^{n-k}\qquad\because f(n)*g(n)=\sum_{k=0}^{n}f(k)g(n-k)$$ $$=\sum_{k=0}^{n}\left(\frac{1}{2}\right)^k\left(\frac{1}{4}\right)^n\left(\frac{1}{4}\right)^{-k}$$ $$=\left(\frac{1}{4}\right)^n\sum_{k=0}^{n}\left(\frac{1}{2}\right)^k(4)^k=\left(\frac{1}{4}\right)^n\sum_{k=0}^{n}\left(\frac{4}{2}\right)^k=\left(\frac{1}{4}\right)^n\sum_{k=0}^{n}(2)^k$$ $$=\left(\frac{1}{4}\right)^n\left[1+2+2^2+2^3+\ldots+2^n\right]$$ $$=\left(\frac{1}{4}\right)^n\left[\frac{2^{n+1}-1}{2-1}\right]\qquad\because 1+a+a^2+a^3+\ldots+a^n=\frac{a^{n+1}-1}{a-1}$$ $$\boxed{Z^{-1}\left[\frac{8z^2}{(2z-1)(4z+1)}\right]=\left(\frac{1}{4}\right)^n\left[2^{n+1}-1\right]}$$
5.	**Using convolution theorem find** $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]$

	Solution: Given $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]=?$ By convolution theorem $Z^{-1}\left[F(z)\cdot G(z)\right]=Z^{-1}\left[F(z)\right]*Z^{-1}\left[G(z)\right]$ $Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]=Z^{-1}\left[\frac{z}{z-1}\cdot\frac{z}{z-3}\right]$ $=Z^{-1}\left[\frac{z}{z-1}\right]*Z^{-1}\left[\frac{z}{z-3}\right]$ $=(1)^n*(3)^n$ $=\sum_{k=0}^{n}(1)^k(3)^{n-k} \qquad \because f(n)*g(n)=\sum_{k=0}^{n}f(k)g(n-k)$ $=\sum_{k=0}^{n}1^k3^{-k}3^n$ $=3^n\sum_{k=0}^{n}\left(\frac{1}{3}\right)^k$ $=3^n\left[1+\left(\frac{1}{3}\right)+\left(\frac{1}{3}\right)^2+\left(\frac{1}{3}\right)^3+\ldots+\left(\frac{1}{3}\right)^n\right]$ $=3^n\left[\frac{\left(\frac{1}{3}\right)^{n+1}-1}{\frac{1}{3}-1}\right]=3^n\left[\frac{\frac{1^{n+1}}{3^{n+1}}-1}{\frac{1}{3}-1}\right]=3^n\left[\frac{\frac{1^{n+1}-3^{n+1}}{3^{n+1}}}{\frac{1-3}{3}}\right]$ $=3^n\left[\frac{1-3^{n+1}}{3^{n+1}}\times\frac{3}{1-3}\right]=\cancel{3^n}\left[\frac{3^{n+1}-3^{n+1}}{\cancel{3^n}\,\cancel{3}}\times\frac{\cancel{3}}{-2}\right]$ $=\frac{-1}{2}\left[1-3^{n+1}\right]$ $\boxed{Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right]==\frac{-1}{2}\left[1-3^{n+1}\right]}$