3.2 CONVOLUTION INTEGRAL

The response of a continuous-time LTI system can be computed by convolution of the impulse response of the system with the input signal, using a convolution integral, rather than a sum.

The response to the input signal $x(t)$ can be written as a convolution integral:

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

or it can be expressed symbolically

$$y(t) = x(t) * h(t)$$

Calculation of convolution integral

The output $y(t)$ is a weighted integral of the input, where the weight on $x(\tau)$ is $h(t - \tau)$ To evaluate this integral for a specific value of t ,

· First obtain the signal $h(t - \tau)$ (regarded as a function of τ with t fixed) from $h(\tau)$ by a reflection about the origin and a shift to the right by t if $t > 0$ or a shift to the left by $|t|$ is $t < 0$.

· Then multiply together the signals $x(\tau)$ and $h(t - \tau)$.

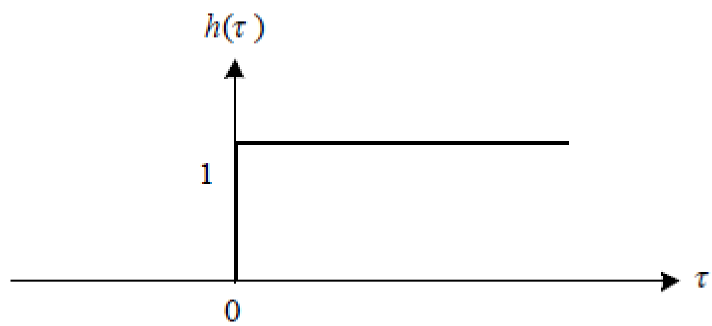
· $y(t)$ is obtained by integrating the resulting product from $\tau = -\infty$ to $\tau = +\infty$

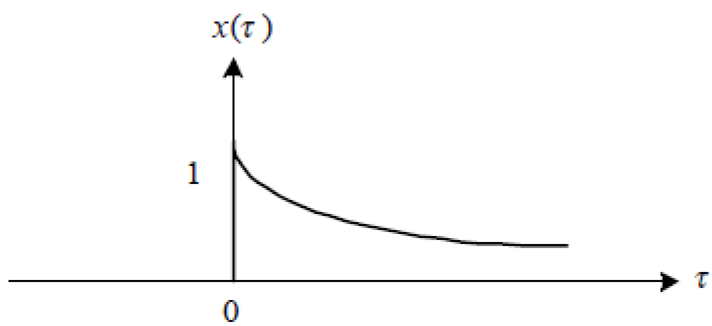
Example: Let $x(t)$ be the input to an LTI system with unit impulse response $h(t)$, where

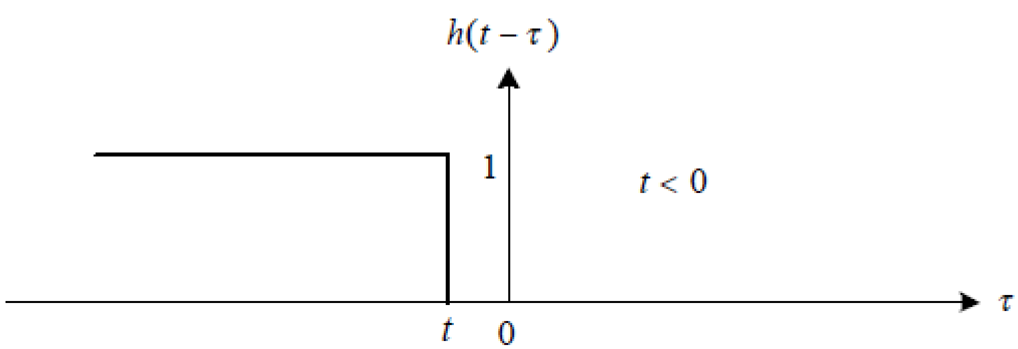
$$x(t) = e^{-at}u(t),\ a > 0 \text{ and } h(t) = u(t).$$

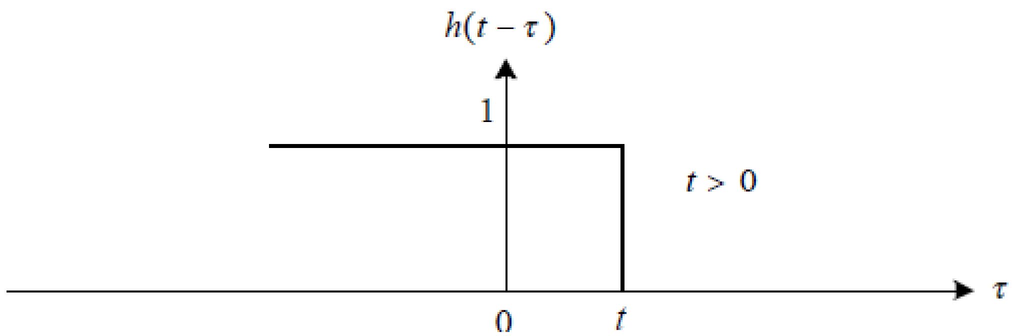
Solution:

Step1: The functions $h(\tau), x(\tau)$ and $h(t-\tau)$ are depicted









Step 2: From the figure we can see that for $t < 0$, the product of the product $x(\tau)$ and $h(t-\tau)$ is zero, and consequently, $y(t)$ is zero. For $t > 0$

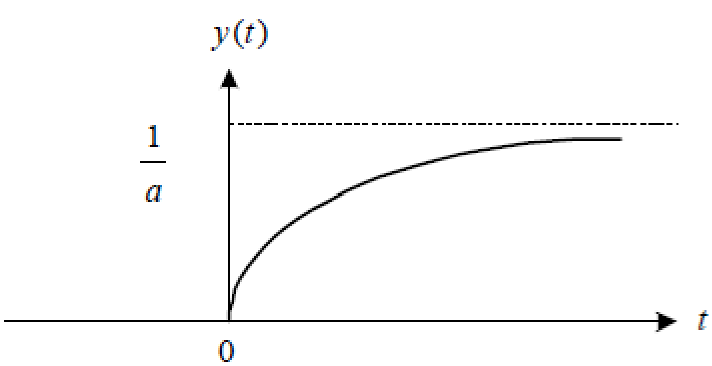
$$x(\tau)h(t-\tau) = \begin{cases} e^{-at}, & t > 0 \\ 0, & otherwise \end{cases}$$

Step 3: Compute $y(t)$ by integrating the product for $t > 0$

$$y(t) = \int_0^t e^{-a\tau} d\tau = -\frac{1}{a} e^{-a\tau}\Big|_0^t = \frac{1}{a}(1 - e^{-at}).$$

The output of $y(t)$ for all t is

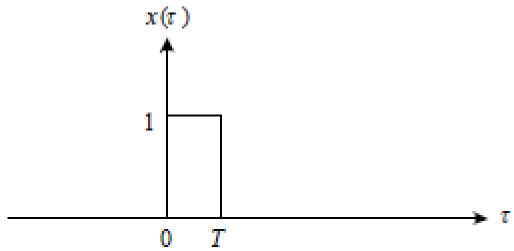
$y(t) = \frac{1}{a}(1 - e^{-at})u(t)$, and is shown in figure below.

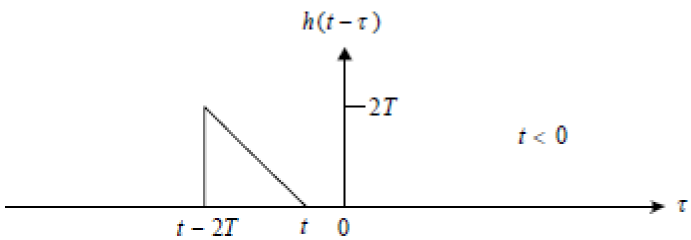


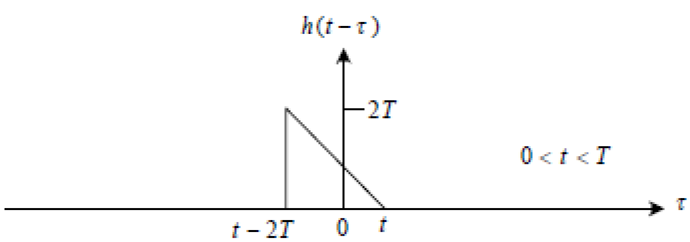
Example: Compute the convolution of the two signals below:

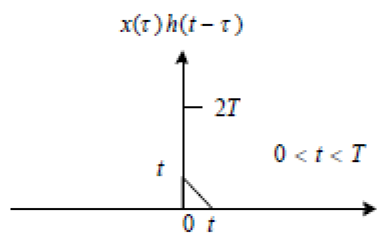
$$x(t) = \begin{cases} 1, & 0 < t < T \\ 0, & otherwise \end{cases} \text{ and } h(t) = \begin{cases} t, & 0 < t < 2T \\ 0, & otherwise \end{cases}$$

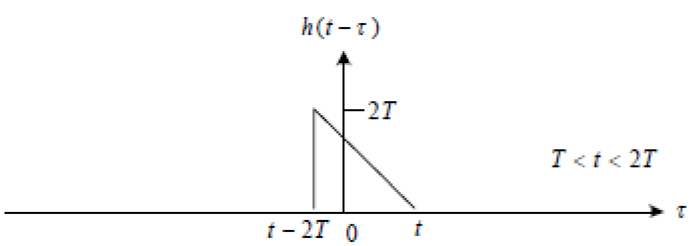
Solution:

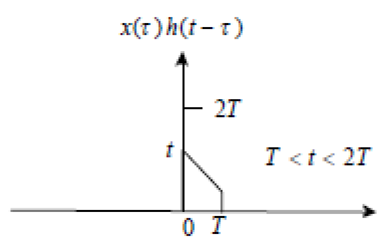


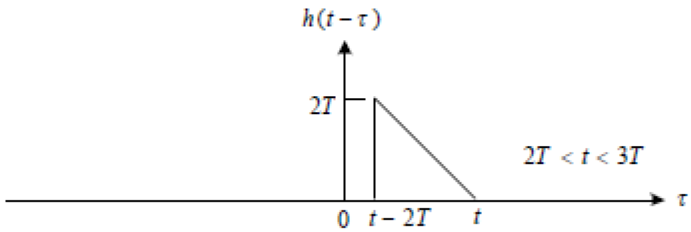


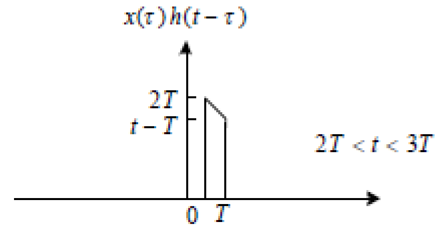


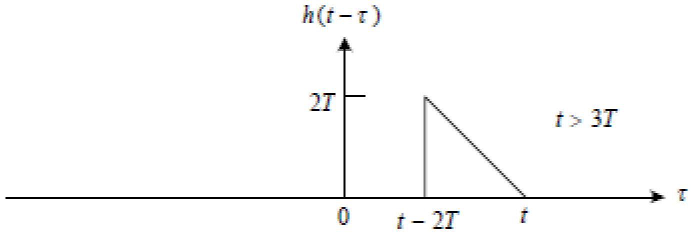












$$y(t) = \begin{cases} 0, & t < 0 \\ \frac{1}{2}t^2, & 0 < t < T \\ Tt - \frac{1}{2}T^2, & T < t < 2T \\ -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, & 2T < t < 3T \\ 0, & t > 3T \end{cases}$$