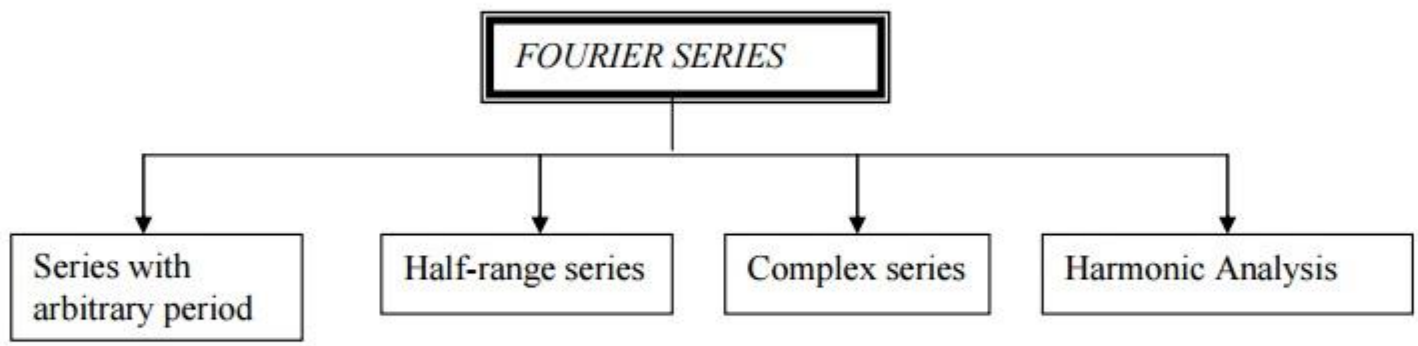
UNIT II – FOURIER SERIES

2.1 INTRODUTION OF FOURIER SERIES

A Fourier series of a periodic function consists of a sum of sine and cosine terms. Sines and cosines are the most fundamental periodic functions.The Fourier series is named after the French Mathematician and Physicist Jacques Fourier (1768 –1830). Fourier series has its application in problems pertaining to Heat conduction, acoustics, etc. The subject matter may be divided into the following sub topics.



Convergence of Fourier Series:

- At a continuous point x = a, Fourier series converges to f(a)
- At end point c or c+2*l* in (c, c+2*l*), Fourier series converges to $\frac{f(c)+f(c+2\ell)}{2}$
- At a discontinuous point x = a, Fourier series converges to $\frac{f(a-)+f(a+)}{2}$

Fourier series in an interval of length 2ℓ

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right)$$

Fourier series of f(x) in $(0,2\ell)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right)$$

$$a_0 = \frac{1}{\ell}\int_0^{2\ell} f(x)\,dx$$

$$a_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

$$b_n = \frac{1}{\ell}\int_0^{2\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

Fourier series of f(x) in $(-\ell,\ell)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\ell} + b_n \sin\frac{n\pi x}{\ell}\right)$$

$$a_0 = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\,dx$$

$$a_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

$$b_n = \frac{1}{\ell}\int_{-\ell}^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

Even Function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\ell}\right)$$

$$a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)\,dx$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

$$b_n = 0$$

Odd Function

$$f(x) = \sum_{n=1}^{\infty}\left(b_n \sin\frac{n\pi x}{\ell}\right)$$

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

Fourier series in the Interval of length 2π

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

Fourier Series of f(x) in (0,2π)

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx\,dx$$

Fourier Series of f(x) in (-π, π)

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right)$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$$

Even Function

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx\right)$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx$$

$$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx$$

$$b_n=0$$

Odd Function

$$f(x)=\sum_{n=1}^{\infty}\left(b_n\sin nx\right)$$

$$a_0=0$$

$$a_n=0$$

$$b_n=\frac{2}{\pi}\int_0^{\pi}f(x)\sin nx\,dx$$

Half Range Fourier series

Fourier Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{\ell}\right)$$

$$a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)\,dx$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x)\cos\frac{n\pi x}{\ell}\,dx$$

Fourier Sine Series

$$f(x) = \sum_{n=1}^{\infty}\left(b_n \sin\frac{n\pi x}{\ell}\right)$$

$$b_n = \frac{2}{\ell}\int_0^{\ell} f(x)\sin\frac{n\pi x}{\ell}\,dx$$

Convergence of Fourier Cosine series:

- At a continuous point x = a, Fourier cosine series converges to f(a).
- At end point 0 in(0,*l*), Fourier cosine series converges to f(0+)
- At end point *l* in(0,*l*), Fourier cosine series converges to f(*l*-)

Convergence of Fourier Sine series:

- At a continuous point x = a, Fourier Sine series converges to f(a).
- At both end points Fourier Sine series converges to 0.

<u>Harmonic Analysis:</u>

$$a_0 = 2\left[\frac{\sum y}{N}\right], \quad a_n = 2\left[\frac{\sum y\cos\left(\frac{n\pi x}{\ell}\right)}{N}\right], \quad b_n = 2\left[\frac{\sum y\sin\left(\frac{n\pi x}{\ell}\right)}{N}\right]$$

Parseval's Theorem:

If $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{\ell} + b_n\sin\frac{n\pi x}{\ell}\right)$ is the Fourier series of f(x) in (c, c+2*l*),

Then $\overline{y}^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ (or) $\frac{1}{2\ell}\int_c^{c+2\ell}[f(x)]^2\,dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$

Root Mean Square Value:

$\bar{y}^2$ is the effective value (or) Root Mean square (RMS) value of the function y = f(x), which is given by

$$\bar{y} = \sqrt{\frac{\int_{c}^{c+2\ell}[f(x)]^2\,dx}{2\ell}}$$

Some Important Results:

1. Sin nπ =0 for all integer values of n
2. Cos nπ= $(-1)^n$ for all integer values of n
3. Cos2nπ=1 for all integer values of n
4. Sin2nπ = 0 for all integer values of n
5. If f(–x) = f(x) then f(x) is even and If f(–x) = – f(x) then f(x) is odd.

6. $f(x) = \begin{cases} \varphi_1(x) & (-\ell, 0) \\ \varphi_2(x) & (0, \ell) \end{cases}$ is even if either $\varphi_1(-x) = \varphi_2(x)$ or $\varphi_2(-x) = \varphi_1(x)$

7. $f(x) = \begin{cases} \varphi_1(x) & (-\ell, 0) \\ \varphi_2(x) & (0, \ell) \end{cases}$ is odd if either $\varphi_1(-x) = -\varphi_2(x)$ or $\varphi_2(-x) = -\varphi_1(x)$

8. $|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$

9. $\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

10. $\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$

11. $\int u\,dv = uv_1 - u'v_2 + u''v_3 - \ldots\ldots$ Where

$u' = \frac{du}{dx},\ u'' = \frac{d^2u}{dx^2}, \ldots\ldots v_1 = \int dv,\ \ v_2 = \int v_1 dx,\ \ v_3 = \int v_2 dx \ldots\ldots\ldots$