DEPARTMENT OF MATHEMATICS

UNIT I - PARTIAL DIFFERENTIAL EQUATIONS

1.1 FORMATION OF PDE BY ELIMINATING ARBITRARY CONSTANTS & FUNCTIONS

Notations: If $z = f(x,y)$ then

$$p=\frac{\partial z}{\partial x}\ ;\ q=\frac{\partial z}{\partial y}\ ;\ r=\frac{\partial^2 z}{\partial x^2}\ ;\ s=\frac{\partial^2 z}{\partial x \partial y}\ ;\ t=\frac{\partial^2 z}{\partial y^2}$$

Formation of PDE by eliminating arbitrary constants:

Let the given equation be $z = f(x,y,a,b) \quad -----(1)$

Step 1: Differentiating (1) partially with respect to x

$$\frac{\partial z}{\partial x} = p = f'(x,y,a,b) \quad ----- (2)$$

Step 2: Differentiating (1) partially with respect to y

$$\frac{\partial z}{\partial y} = q = f'(x,y,a,b) \quad ----- (3)$$

Step 3: Eliminate a & b from (1) using (2) & (3)

1. **Obtain partial differential equation by eliminating arbitrary constant 'a' and 'b' from** $z=(x-a)^2+(y-b)^2$

Solution:

Given $z=(x-a)^2+(y-b)^2 \quad ----(1)$

Diff Partially w.r.t x

$$\frac{\partial z}{\partial x} = 2(x-a)+0$$

$$p = 2(x-a) \quad ----(2)$$

	Diff Partially w.r.t y $\frac{\partial z}{\partial y} = 0 + 2(y-b)$ $q = 2(y-b) \; ----(3)$ Eliminate $a \,\&\, b$ from (1) using (2) & (3) $(2) \Rightarrow (x-a) = \frac{p}{2} \; ----(4)$ $(3) \Rightarrow y-b = \frac{q}{2} \; ----(5)$ Sub (4) & (5) in (1) $(1) \Rightarrow z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$ The required the PDE is $\boxed{p^2 + q^2 = 4z}$
2.	**Form the partial differential equation by eliminating the arbitrary constants 'a' & 'b' from** $z = (x^2+a)(y^2+b)$. **Solution:** **Given** $z = (x^2+a)(y^2+b) \; ----(1)$ Diff Partially w.r.t x $\frac{\partial z}{\partial x} = p = 2x(y^2+b) \; ----(2)$ Diff Partially w.r.t y $\frac{\partial z}{\partial y} = q = 2y(x^2+a) \; ---- \; (3)$ Eliminate $a \,\&\, b$ from (1) using (2) & (3) $(2) \Rightarrow (y^2+b) = \frac{p}{2x} ----(4)$

	$(3) \Rightarrow x^2 + b = \frac{q}{2y} ----(5)$ Sub (4) & (5) in (1) $(1) \Rightarrow z = \left(\frac{p}{2x}\right)\left(\frac{q}{2y}\right)$ The required the PDE is $\boxed{4xyz = pq}$
3.	**Find the PDE of all planes having equal intercepts on the x and y axis.** **Solution:** The intercept form of the plane equation is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ Given that equal intercepts on the x & y axis $\Rightarrow$ $a = b$ $\therefore \frac{x}{a} + \frac{y}{a} + \frac{z}{c} = 1 -----(1)$ Diff Partially w.r.t x $\frac{1}{a} + 0 + \frac{1}{c}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{1}{a} = \frac{-1}{c} p ----(2)$ Diff Partially w.r.t y $0 + \frac{1}{a}(1) + \frac{1}{c}\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{1}{a} = \frac{-1}{c} q ----(3)$ From (2) & (3) $\frac{-1}{c} p = \frac{-1}{c} q$ The required the PDE is $\boxed{p = q}$
4.	**Obtain the partial differential equation by eliminating arbitrary constants 'a' and ' b' from** $(x-a)^2 + (y-b)^2 + z^2 = r^2$ **Solution:** $(x-a)^2 + (y-b)^2 + z^2 = 1 ----(1)$ - (1) Diff Partially w.r.t x

$$2(x-a)(1-0)+0+2z\frac{\partial z}{\partial x}=0$$

$$\Rightarrow 2(x-a)+2zp=0 \;-----\; (2)$$

Diff Partially w.r.t y

$$0+2(y-b)(1-0)+2z\frac{\partial z}{\partial y}=0$$

$$\Rightarrow 2(y-b)+2zq=0 \;-----\; (3)$$

Eliminate $a \,\&\, b$ from (1) using (2) & (3)

$$(2)\Rightarrow x-a=-zp \;-----\; (4)$$

$$(3)\Rightarrow y-b=-zq \;-----\; (5)$$

Sub (4) & (5) in (1)

$$(-zp)^2+(-zq)^2+z^2=1$$

The required PDE is $\boxed{z^2\left(p^2+q^2+1\right)=1}$

Formation of PDE by eliminating arbitrary functions:

1. **Eliminate the arbitrary function *f* from $z=f\left(\frac{y}{x}\right)$ and form the PDE.**

Solution:

$$z=f\left(\frac{y}{x}\right)----(1)$$

Diff Partially w.r.t x

$$\frac{\partial z}{\partial x}=p=f'\left(\frac{y}{x}\right)\times\left(\frac{-y}{x^2}\right) \;\Rightarrow f'\left(\frac{y}{x}\right)=\frac{-px^2}{y}----(2)$$

Diff Partially w.r.t x

$$\frac{\partial z}{\partial y}=q=f'\left(\frac{y}{x}\right)\times\left(\frac{1}{x}\right) \;----(3)$$

From (1) & (2) $\frac{p}{q}=\frac{-y}{x} \Rightarrow \boxed{px+qy=0}$

2.	**Form the partial differential equation by eliminating *f* from** $z = x^2 + 2f\left(\frac{1}{y} + \log x\right)$**.** **Solution:** **Given** $z = x^2 + 2f\left(\frac{1}{y} + \log x\right) -----(1)$ Differentiate (1) partially w.r.t x $\frac{\partial z}{\partial x} = 2x + 2f'\left(\frac{1}{y} + \log x\right)\left(0 + \frac{1}{x}\right)$ $p = 2x + 2f'\left(\frac{1}{y} + \log x\right)\left(\frac{1}{x}\right) \Rightarrow f'\left(\frac{1}{y} + \log x\right) = (p - 2x)\frac{x}{2} ----(2)$ $\frac{\partial z}{\partial y} = 2f'\left(\frac{1}{y} + \log x\right)\left(\frac{-1}{y^2} + 0\right)$ $q = \frac{-2}{y^2} f'\left(\frac{1}{y} + \log x\right) \Rightarrow f'\left(\frac{1}{y} + \log x\right) = \frac{-qy^2}{2} -----(3)$ Eliminating f' from (2) & (3) $(p - 2x)\frac{x}{2} = \frac{-qy^2}{2} \Rightarrow (px - 2x^2) = -qy^2$ $\Rightarrow \boxed{px + qy^2 = 2x^2}$

Formation of PDE by eliminating *f* from $f(u,v) = 0$ ------(1)

Method 1:

The required PDE of (1) is $\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$

Method 2:

The required PDE is $Pp + Qq = R$

Where

$P=\begin{vmatrix} u_y & v_y \\ u_z & v_z \end{vmatrix}$; $Q=\begin{vmatrix} u_z & v_z \\ u_x & v_x \end{vmatrix}$; $R=\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}$	
1.	**Form the PDE from** $\phi\left(ax+by+cz, x^2+y^2+z^2\right)=0$ **Solution:** Given $\phi\left(ax+by+cz, x^2+y^2+z^2\right)=0$ This is of the form $f(u,v)=0$ where $u=ax+by+cz$ & $v=x^2+y^2+z^2$ The required PDE of (1) is $\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}=0$ $\begin{vmatrix} p & q & -1 \\ a & b & c \\ 2x & 2y & 2z \end{vmatrix}=0$ $\Rightarrow p(2bz-2cy)-q(2az-2cx)+1(2az-2cx)=0$ $\div 2 \Rightarrow \boxed{(bz-cy)p+(cx-az)q+(az-cx)=0}$
2.	**Form the PDE from** $\phi\left(x^2+y^2+z^2, xyz\right)=0$ **Solution:** Given $\phi\left(x^2+y^2+z^2, xyz\right)=0$ This is of the form $f(u,v)=0$ where $u=x^2+y^2+z^2$ & $v=xyz$ The required PDE of (1) is $\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix}=0$ $\begin{vmatrix} p & q & -1 \\ 2x & 2y & 2z \\ yz & xz & xy \end{vmatrix}=0$ $\Rightarrow p(2xy^2-2xz^2)-q(2x^2y-2yz^2)+1(2x^2z-2y^2z)=0$

	$\div 2 \Rightarrow \boxed{x(y^2-z^2)p + y(z^2-x^2) + z(x^2-y^2) = 0}$
3.	**Form the PDE from** $\phi\left(\frac{y}{x}, x^2+y^2+z^2\right) = 0$ **Solution:** Given $\phi\left(\frac{y}{x}, x^2+y^2+z^2\right) = 0$ This is of the form $f(u,v) = 0$ where $u = \frac{y}{x}$ & $v = x^2+y^2+z^2$ The required PDE of (1) is $\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$ $\begin{vmatrix} p & q & -1 \\ \frac{-y}{x^2} & \frac{1}{x} & 0 \\ 2x & 2y & 2z \end{vmatrix} = 0$ $\Rightarrow p\left(\frac{2z}{x} - 0\right) - q\left(\frac{-2yz}{x^2} - 0\right) + 1\left(\frac{-2y^2}{x^2} - \frac{2x}{x}\right) = 0$ $\Rightarrow \frac{2zp}{x} + \frac{2yzq}{x^2} - 2\left(\frac{y^2}{x^2} + 1\right) = 0$ $\div 2 \Rightarrow \frac{zp}{x} + \frac{yzq}{x^2} - \left(\frac{y^2+x^2}{x^2}\right) = 0$ $\Rightarrow \frac{xzp + yzq - \left(y^2+x^2\right)}{x^2} = 0$ $\boxed{xzp + yzq - \left(y^2+x^2\right) = 0}$