UNIT I - PARTIAL DIFFERENTIAL EQUATIONS

1.2 SOLUTIONS OF STANDARD TYPES OF FIRST ORDER PDE

Solutions of standard types of First order PDE's:

Different solutions of PDE:

Complete Integral (or) Complete Solution:

If the number of arbitrary constants is equal to number of independent variables, then the solution is called Complete integral.

Singular Integral (or) Singular Solution:

Consider a PDE of first order as $f(x, y, z, p, q) = 0 \quad -----(1)$

It's complete integral may be, $f(x, y, z, a, b) = 0 \quad -----(2)$

Diff (2) partially with respect to $a \& b$ respectively,

$$\frac{\partial g}{\partial a} = 0 \quad ------(3)$$

$$\frac{\partial g}{\partial b} = 0 \quad ------(4)$$

Eliminating $a \& b$ from (3) & (4) will get the Singular integral.

General Integral (or) Complete solution:

A Solution which contains number of arbitrary functions is equal to the order of the given PDE.

(or) A solution which contains the maximum possible number of arbitrary functions.

Type I:

Equations of the form $f(p, q) = 0 \quad -----(1)$

To find Complete Integral:

Let the complete solution of (1) is $z = ax + by + c \quad ----(2)$

Let $p = a$ & $q = b$ in (1)

$f(a,b) = 0$ and represent $b = \phi(a)$

$\therefore (1) \Rightarrow z = ax + \phi(a)y + c \quad ----(3)$

To Find Singular Integral:

Diff (3) partially with respect to c

$0 = 1$ which is impossible

There is no singular integral for this type.

To find General integral:

Put $c = g(a)$ in (3)

$(3) \Rightarrow z = ax + \phi(a)y + g(a) \quad ----(4)$

Diff (4) partially with respect to a

$0 = x(1) + \phi'(a)y + g'(a) \quad ----(5)$

Eliminating a from (4) & (5) we get general integral.

1. **Find the complete integral of** $p + q = pq$

Solution:

Given $p + q = pq \quad ----(1)$

This of the form $f(p,q) = 0$

To find Complete Integral:

Let the complete solution of (1) is $z = ax + by + c \quad ----(2)$

Let $p = a$ & $q = b$ in (1)

$(1) \Rightarrow a + b = ab \Rightarrow a + b - ab = 0 \Rightarrow b = \dfrac{a}{a-1}$

	Sub b in (2) $$\boxed{z = ax + \left(\frac{a}{a-1}\right)y + c}$$ This is the required complete integral.
2.	**Find the complete integral of** $p + q = 1$ **Solution:** Given $p + q = 1 \quad ----(1)$ This of the form $f(p,q) = 0$ **To find Complete Integral:** Let the complete solution of (1) is $z = ax + by + c \quad ----(2)$ Let $p = a$ & $q = b$ in (1) $(1) \Rightarrow a + b = 1 \Rightarrow b = 1 - a$ Sub b in (2) $$\boxed{z = ax + (1-a)y + c}$$ This is the required complete integral.
3.	**Solve** $\sqrt{p} + \sqrt{q} = 1$ **Solution:** Given $\sqrt{p} + \sqrt{q} = 1 \quad ----(1)$ This of the form $f(p,q) = 0$ **To find Complete Integral:** Let the complete solution of (1) is $z = ax + by + c \quad ----(2)$ Let $p = a$ & $q = b$ in (1) $(1) \Rightarrow \sqrt{a} + \sqrt{b} = 1 \Rightarrow \sqrt{b} = 1 - \sqrt{a} \quad \Rightarrow b = \left(1 - \sqrt{a}\right)^2$

Sub b in (2)

$$\boxed{z = ax + \left(1-\sqrt{a}\right)^2 y + c} \quad ----(3)$$

This is the required complete integral.

To Find Singular Integral:

Diff (3) partially with respect to c

$0 = 1$ which is impossible

There is no singular integral for this type.

To find General integral:

Put $c = f(a)$ in (3)

$$(3) \Rightarrow \quad z = ax + \left(1-\sqrt{a}\right)^2 y + f(a) \quad ----(4)$$

Diff. (4) partially with respect to a

$$0 = x(1) + 2\left(1-\sqrt{a}\right)\frac{1}{2\sqrt{a}}\, y + f'(a) \quad ----(5)$$

Eliminate a from (4) & (5) we get the general integral.

Type II:

Equations of the form $z = px + qy + f(p,q) \quad ------(1)$

To find Complete Integral:

Put $p = a$ & $q = b$ in (1)

$$\therefore (1) \Rightarrow z = ax + by + f(\text{a,b}) \quad ------(2)$$

To Find Singular Integral:

Diff (2) partially with respect to a

$$0 = x(1) + 0 + f'(a,b) \quad ----(3)$$

Diff (2) partially with respect to b

$$0 = 0 + y(1) + f'(a,b) \quad ----(4)$$

Eliminating a & b from (2) using (3) &(4), we get the singular integral. **To find General integral:** Put $b=\phi(a)$ in (2) $(2)\Rightarrow\ z=ax+\phi(a)y+g(\text{a})\ ------(5)$ Diff (4) partially with respect to a $0=x(1)+\phi'(a)y+g'(a)\ ----(6)$ Eliminating a from (5) & (6) we get General integral.	
1.	**Solve** $z=px+qy+p^2q^2$ **Solution:** Given $z=px+qy+p^2q^2\ -----(1)$ Equations of the form $z=px+qy+f(p,q)$ To find Complete Integral: Put $p=a$ & $q=b$ in (1) $\therefore$ (1) $\Rightarrow \boxed{z=ax+by+a^2b^2}------(2)$ This is the required complete integral **To Find Singular Integral:** Diff (2) partially with respect to a $0=x(1)+0+2ab^2\ \Rightarrow x+2ab^2=0\ \Rightarrow x=-2ab^2----(3)$ Diff (2) partially with respect to b $0=0+y(1)+0+2a^2b\ \Rightarrow y+2a^2b=0\ \Rightarrow y=-2a^2b----(4)$ Eliminating a & b from (2) using (3) &(4) $(3)\Rightarrow\ \dfrac{x}{b}=-2ab----(5)$

$$(4) \Rightarrow \frac{y}{a} = -2ab ----(6)$$

From (5) & (6)

$$\frac{x}{b} = \frac{y}{a} = k \text{ (say)}$$

$$\Rightarrow \frac{x}{b} = k \;\; \& \;\; \frac{y}{a} = k$$

$$\Rightarrow b = \frac{x}{k} \;\; \& \;\; a = \frac{y}{k} \;\; ----(7)$$

Sub a & b in (2)

$$(2) \Rightarrow z = \frac{y}{k}x + \frac{x}{k}y + \frac{y^2}{k^2}\frac{x^2}{k^2}$$

$$z = \frac{xy}{k} + \frac{xy}{k} + \frac{x^2y^2}{k^4}$$

$$z = \frac{2xy}{k} + \frac{x^2y^2}{k^4} \;\; ----(8)$$

<u>To find k</u>

Sub (7) in (3) (or) (4)

$$(3) \Rightarrow x = -2\frac{y}{k}\frac{x^2}{k^2} \;\; \Rightarrow x = \frac{-2x^2y}{k^3} \;\; \Rightarrow k^3 = -2xy$$

$$(8) \Rightarrow \;\; z = \frac{2xy}{k} + \frac{x^2y^2}{k(-2xy)} \;\; \Rightarrow z = \frac{2xy}{k} - \frac{xy}{2k}$$

$$z = \frac{4xy - xy}{2k} \;\; \Rightarrow z = \frac{3xy}{2k} \;\; \Rightarrow z^3 = \frac{27x^3y^3}{8k^3} \;\; \Rightarrow z^3 = \frac{27x^3y^3}{8(-2xy)}$$

$$\boxed{16z^3 = -27x^2y^2}$$

This is the required singular integral.

To find General integral:

	Put $b=\phi(a)$ in (2) $(2)\Rightarrow\ z=ax+f(a)y+a^2\left[f(a)\right]^2------(9)$ Diff (9) partially with respect to a $0=x(1)+f'(a)y+2f(a)f'(a)\quad ----(10)$ Eliminating a from (9) & (10) we get General integral.
2.	**Find the singular integral of** $z=px+qy+p^2+pq+q^2$ **Solution:** Given $z=px+qy+p^2+pq+q^2\ -----(1)$ Equations of the form $z=px+qy+f(p,q)$ To find Complete Integral: Put $p=a\ \ \&\ q=b$ in (1) $\therefore (1)\Rightarrow \boxed{z=ax+by+a^2+ab+b^2}------(2)$ This is the required complete integral **To Find Singular Integral:** Diff (2) partially with respect to a $0=x(1)+0+2a+b+0\ \Rightarrow 2a+b=-x\quad ----(3)$ Diff (2) partially with respect to b $0=0+y(1)+0+a+2b\ \Rightarrow\ a+2b=-y\ ----(4)$ Eliminating a & b from (2) using (3) &(4) $(4)\times 2\Rightarrow 2a+4b=-2y\ -----(5)$ $(3)-(5)\Rightarrow\ -3b=-x+2y\ \ \Rightarrow \boxed{b=\frac{x-2y}{3}}$ Sub the value of b in (3)

	$2a+b=-x \Rightarrow 2a=-x-b \Rightarrow 2a=-x-\left(\frac{x-2y}{3}\right)$ $2a=\frac{-3x-x+2y}{3} \Rightarrow 6a=-4x+2y \Rightarrow \boxed{a=\frac{y-2x}{3}}$ Sub the value of a & b in (2) $z=\left(\frac{y-2x}{3}\right)x+\left(\frac{x-2y}{3}\right)y+\left(\frac{y-2x}{3}\right)^2+\left(\frac{y-2x}{3}\right)\left(\frac{x-2y}{3}\right)+\left(\frac{x-2y}{3}\right)^2$ $z=\frac{xy-2x^2}{3}+\frac{xy-2y^2}{3}+\frac{y^2-4xy+4y^2}{9}+\frac{xy-2y^2-2x^2+4xy}{9}+\frac{x^2-4xy+4y^2}{9}$ $z=\frac{3xy-3x^2+3xy-6y^2+y^2-4xy+4y^2+xy-2y^2-2x^2+4xy+x^2-4xy+4y^2}{9}$ $\boxed{9z=-4x^2+y^2+xy}$
3.	**Solve** $z=px+qy+\sqrt{p^2+q^2+1}$ Solution: Given $z=px+qy+\sqrt{p^2+q^2+1}$ -----(1) To find Complete Integral: Put $p=a$ & $q=b$ in (1) $(1) \Rightarrow \boxed{z=ax+by+\sqrt{a^2+b^2+1}}$ ----(2) This is required complete integral. **To Find Singular Integral:** Diff (2) partially with respect to a $0=x(1)+0+\frac{1}{2\sqrt{a^2+b^2+1}}(2a) \Rightarrow \boxed{x=\frac{-a}{\sqrt{a^2+b^2+1}}}$ ----(3)

Diff (2) partially with respect to b

$$0 = 0 + y(1) + \frac{1}{2\sqrt{a^2+b^2+1}}(2b) \Rightarrow \boxed{y = \frac{-b}{\sqrt{a^2+b^2+1}}} ----(4)$$

Eliminating a & b from (2) using (3) &(4)

$$(3)^2 + (4)^2 \Rightarrow x^2 + y^2 = \left(\frac{-a}{\sqrt{a^2+b^2+1}}\right)^2 + \left(\frac{-b}{\sqrt{a^2+b^2+1}}\right)^2$$

$$x^2 + y^2 = \frac{a^2}{a^2+b^2+1} + \frac{b^2}{a^2+b^2+1}$$

$$x^2 + y^2 = \frac{a^2+b^2}{a^2+b^2+1}$$

$$1-(x^2+y^2) = 1 - \frac{a^2+b^2}{a^2+b^2+1}$$

$$1 - x^2 - y^2 = \frac{a^2+b^2+1-a^2-b^2}{a^2+b^2+1}$$

$$1 - x^2 - y^2 = \frac{1}{a^2+b^2+1}$$

$$\Rightarrow 1 + a^2 + b^2 = \frac{1}{1-x^2-y^2}$$

Taking square root on both sides

$$\Rightarrow \boxed{\sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}} \quad ----(5)$$

Sub (5) in (2) and (3)

$$(3) \Rightarrow x = \frac{-a}{\frac{1}{\sqrt{1-x^2-y^2}}} \Rightarrow x = -a\sqrt{1-x^2-y^2} \Rightarrow \boxed{a = \frac{-x}{\sqrt{1-x^2-y^2}}}$$

	$(4) \Rightarrow y = \dfrac{-b}{\dfrac{1}{\sqrt{1-x^2-y^2}}} \Rightarrow y = -b\sqrt{1-x^2-y^2} \Rightarrow \boxed{b = \dfrac{-y}{\sqrt{1-x^2-y^2}}}$ Sub (5), a & b in (2) $(2) \Rightarrow z = \left(\dfrac{-x}{\sqrt{1-x^2-y^2}}\right)x + \left(\dfrac{-y}{\sqrt{1-x^2-y^2}}\right)y + \dfrac{1}{\sqrt{1-x^2-y^2}}$ $\Rightarrow z = \dfrac{-x^2}{\sqrt{1-x^2-y^2}} + \dfrac{-y^2}{\sqrt{1-x^2-y^2}} + \dfrac{1}{\sqrt{1-x^2-y^2}}$ $\Rightarrow z = \dfrac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$ $\Rightarrow z = \sqrt{1-x^2-y^2}$ Squaring on both sides $z^2 = 1-x^2-y^2 \Rightarrow \boxed{x^2+y^2+z^2=1}$
4.	**Find the singular integral of** $z = px+qy+p^2-q^2$ **Solution:** Given $z = px+qy+p^2-q^2$ ------(1) Equations of the form $z = px+qy+f(p,q)$ To find Complete Integral: Put $p=a$ & $q=b$ in (1) $\therefore (1) \Rightarrow \boxed{z = ax+by+a^2-b^2}$ -------(2) This is the required complete integral **To Find Singular Integral:** Diff (2) partially with respect to a

$$0 = x(1) + 0 + 2a + 0 \Rightarrow a = \frac{-x}{2} \quad ----(3)$$

Diff (2) partially with respect to b

$$0 = 0 + y(1) + 0 - 2b \Rightarrow b = \frac{y}{2} \quad ----(4)$$

Sub a & b in (2)

$$\therefore (2) \Rightarrow z = \left(\frac{-x}{2}\right)x + \left(\frac{y}{2}\right)y + \left(\frac{-x}{2}\right)^2 - \left(\frac{y}{2}\right)^2$$

$$\therefore (2) \Rightarrow z = \frac{-x^2}{2} + \frac{y^2}{2} + \frac{x^2}{4} - \frac{y^2}{4}$$

$$\therefore (2) \Rightarrow z = \frac{-2x^2 + 2y^2 + x^2 - y^2}{4} \Rightarrow \boxed{4z = -x^2 + y^2}$$

This is the required singular integral.

Type III:

Equations of the form $f(\text{z}, p, q) = 0 \;-----(1)$

In this type x & y do not appear explicitly.

To find Complete Integral:

Let the complete solution of (1) is $z = f(x + ay) \;------(2)$

Let $x + ay = u$

(2) $\Rightarrow z = f(\text{u}) \;------(3)$

By total derivative,

$$\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \Rightarrow p = \frac{dz}{dx}(1) \qquad \because u = x + ay \Rightarrow \frac{\partial u}{\partial x} = 1$$

$$\frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} \Rightarrow q = a\frac{dz}{du} \qquad \because u = x + ay \Rightarrow \frac{\partial u}{\partial y} = a$$

Substitute the value of p & q in (1)

$$(1) \Rightarrow f\left(z, \frac{dz}{du}, a\frac{dz}{du}\right) = 0$$

This may be solve by method of separation of variables

Other solutions can obtain as usual.

1. **Solve** $p(1+q) = qz$ **.**

Solution:

Given $p(1+q) = qz$ ----(1)

This is of the form $f(z, p, q) = 0$

To find Complete Integral:

Let the complete solution of (1) is $z = f(x+ay)$ ------(2)

Let $x + ay = u \Rightarrow z = f(u)$

Then $p = \frac{dz}{du}$ & $q = a\frac{dz}{du}$

Substitute the value of p & q in (1)

$$(1) \Rightarrow \frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = a\frac{dz}{du}z$$

$$1 + a\frac{dz}{du} = a\,z$$

$$\frac{dz}{du} = a\,z - 1$$

$$\frac{dz}{a\,z-1} = du$$

Integrating on both sides

$$\int \frac{dz}{a\,z-1} = \int du$$

	$u = \log(az-1) + c$ $\quad\quad\because \int \frac{f'(x)}{f(x)}dx = \log f(x)$ $\boxed{x + ay = \log(az-1) + c}$ This is the required complete integral. Other solutions can be obtained as usual.
2.	**Solve** $z^2 = 1 + p^2 + q^2$ Solution: Given $z^2 = 1 + p^2 + q^2$ $----(1)$ This is of the form $f(z, p, q) = 0$ **To find Complete Integral:** Let the complete solution of (1) is $z = f(x+ay)$ $-------(2)$ Let $x + ay = u \Rightarrow z = f(u)$ Then $p = \frac{dz}{du}$ & $q = a\frac{dz}{du}$ Substitute the value of p & q in (1) $(1) \Rightarrow z^2 = \left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 + 1$ $\Rightarrow \left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 = z^2 - 1$ $\Rightarrow (1+a^2)\left(\frac{dz}{du}\right)^2 = z^2 - 1$ $\Rightarrow \left(\frac{dz}{du}\right)^2 = \frac{z^2-1}{1+a^2}$ Taking square root on both sides

$$\frac{dz}{du}=\frac{\sqrt{z^2-1}}{\sqrt{1+a^2}}$$

$$\Rightarrow \frac{dz}{\sqrt{z^2-1}}=\frac{du}{\sqrt{1+a^2}}$$

Integrating on both sides

$$\cosh^{-1} z=\frac{1}{\sqrt{a^2-1}}u+c \qquad \because \int \frac{dz}{\sqrt{x^2-1}}=\cosh^{-1} x$$

$$\boxed{\cosh^{-1} z=\frac{1}{\sqrt{a^2-1}}(x+ay)+c} \qquad \because u=x+ay$$

This is the required complete integral.

Other solutions can be obtained as usual.

3. **Solve** $p\left(1-q^2\right)=q(1-z)$

Solution:

Given $p\left(1-q^2\right)=q(1-z)$ $----(1)$

This is of the form $f(\text{z}, p, q)=0$

To find Complete Integral:

Let the complete solution of (1) is $z=f(x+ay)$ $------(2)$

Let $x+ay=u \Rightarrow z=f(u)$

Then $p=\frac{dz}{du}$ & $q=a\frac{dz}{du}$

Substitute the value of p & q in (1)

$$(1)\Rightarrow \frac{dz}{du}\left[1-\left(a\frac{dz}{du}\right)^2\right]=a\frac{dz}{du}(1-z)$$

	$1-a+az=a^2\left(\frac{dz}{du}\right)^2$ Taking square root on both sides $a\frac{dz}{du}=\sqrt{1-a+az}$ $\frac{a\ dz}{\sqrt{1-a+az}}=du$ Integrating on both sides $a\left(\frac{2\sqrt{1-a+az}}{a}\right)=u+c \qquad \because \int\frac{1}{\sqrt{ax}}dx=\frac{1}{a}\left(2\sqrt{x}\right)$ $\boxed{2\sqrt{1-a+az}=x+ay+c} \qquad \because u=x+ay$ This is the required complete integral. Other solutions can be obtained as usual.

Type IV:

Equations of the form $f_1(x,p)=f_2(y,q)$ $------(1)$

To find Complete Integral:

Let $f_1(x,p)=f_2(y,q)=a\,(\text{say})$

$\therefore f_1(x,p)=a \quad ; \quad f_2(y,q)=a$

From the above we get $p=f_1(x,a)$; $q=f_2(y,b)$

Substitute the value of p & q in $z=\int pdx+\int qdy$

Integrating we get complete integral

Other solutions can obtain as usual.

1.	**Solve** $p^2+q^2=x^2+y^2$ **Solution:**

Given $p^2+q^2=x^2+y^2$

$$p^2-x^2=y^2-q^2 \quad -----(1)$$

This is of the form $f_1(x,p)=f_2(y,q)$

To find Complete Integral:

Let $p^2-x^2=y^2-q^2=a^2$ (say)

$$\therefore p^2-x^2=a^2 \;\; ; \; y^2-q^2=a^2$$

$$\therefore p^2=a^2+x^2 \;\; ; \; q^2=y^2-a^2$$

$$\boxed{p=\sqrt{a^2+x^2}} \;\; ; \; \boxed{q=\sqrt{y^2-a^2}}$$

Substitute the value of p & q in

$$z=\int\sqrt{x^2+a^2}\,dx+\int\sqrt{y^2-a^2}\,dy$$

$$\boxed{z=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)+\frac{y}{2}\sqrt{y^2-a^2}+\frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right)+c}$$

$$\because 1)\int\sqrt{x^2+a^2}\,dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) \quad 2)\int\sqrt{y^2-a^2}\,dy=\frac{y}{2}\sqrt{y^2-a^2}+\frac{a^2}{2}\cosh^{-1}\left(\frac{y}{a}\right)$$

Integrating we get complete integral

Other solutions can obtain as usual.

2. **Find the complete integral of** $p^2y(1+x^2)=qx^2$

Solution:

Given $p^2y(1+x^2)=qx^2$

$$\frac{p^2(1+x^2)}{x^2}=\frac{q}{y} \quad -----(1)$$

This is of the form $f_1(x,p)=f_2(y,q)$

To find Complete Integral:

	Let $\frac{p^2(1+x^2)}{x^2}=\frac{q}{y}=a\,(\text{say})$ $\frac{p^2(1+x^2)}{x^2}=a\ ;\frac{q}{y}=a$ $p^2=a\frac{x^2}{1+x^2}\quad ;\ q=ay$ $\boxed{p=\frac{\sqrt{a}\ x}{\sqrt{1+x^2}}}\quad ;\ \boxed{q=ay}$ Substitute the value of p & q in $z=\int\frac{\sqrt{a}\ x}{\sqrt{1+x^2}}dx+\int ay\ dy$ let $1+x^2=t\ \Rightarrow 2xdx=dt\ \Rightarrow xdx=\frac{dt}{2}$ $z=\sqrt{a}\int\frac{1}{\sqrt{t}}\frac{dt}{2}+a\frac{y^2}{2}$ $z=\frac{\sqrt{a}}{2}2\sqrt{t}+\frac{ay^2}{2}\qquad \because\int\frac{1}{\sqrt{x}}dx=2\sqrt{x}$ $\boxed{z=\sqrt{a}\sqrt{1+x^2}+\frac{ay^2}{2}+c}$
3.	**Find the complete integral of** $p+q=\sin x+\sin y$ **Solution:** Given $p+q=\sin x+\sin y$ $p-\sin x=\sin y-q\ \ -----(1)$ This is of the form $f_1(x,p)=f_2(y,q)$ **To find Complete Integral:** Let $p-\sin x=\sin y-q=a\,(\text{say})$

	$\therefore p-\sin x=a \quad ; \sin y-q=a$ $\therefore \boxed{p=\sin x+a} \quad ; \boxed{q=\sin y-a}$ Substitute the value of p & q in $z=\int(\sin x+a)\,dx+\int(\sin y-a)\,dy$ $z=\cos x+ax+\cos y-ay+c$ $\boxed{z=\cos x+\cos y-a(x-y)+c}$ This is the required complete integral