3.2 Newton-Euler Method

Newton-Euler for Single Bodies

Evaluating the principle of virtual work (3.13) for a single body results to:

$$0 = \delta W = \int_{\mathcal{B}} \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \mathbf{\Phi} \end{pmatrix}^T \begin{bmatrix} \mathbb{I}_{3\times 3} \\ [\boldsymbol{\rho}]_\times \end{bmatrix} \left(\begin{bmatrix} \mathbb{I}_{3\times 3} & -[\boldsymbol{\rho}]_\times \end{bmatrix} \begin{pmatrix} \mathbf{a}_s \\ \mathbf{\Psi} \end{pmatrix} \mathrm{d}m + [\mathbf{\Omega}]_\times^2 \boldsymbol{\rho} \mathrm{d}m - \mathrm{d}\mathbf{F}_{ext} \right)$$

$$= \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \mathbf{\Phi} \end{pmatrix}^T \int_{\mathcal{B}} \left(\begin{bmatrix} \mathbb{I}_{3\times 3}\mathrm{d}m & [\boldsymbol{\rho}]_\times^T \mathrm{d}m \\ [\boldsymbol{\rho}]_\times \mathrm{d}m & -[\boldsymbol{\rho}]_\times^2 \mathrm{d}m \end{bmatrix} \begin{pmatrix} \mathbf{a}_s \\ \mathbf{\Psi} \end{pmatrix} + \begin{pmatrix} [\mathbf{\Omega}]_\times^2 \boldsymbol{\rho} \mathrm{d}m \\ [\boldsymbol{\rho}]_\times [\mathbf{\Omega}]_\times^2 \boldsymbol{\rho} \mathrm{d}m \end{pmatrix} - \begin{pmatrix} \mathrm{d}\mathbf{F}_{ext} \\ [\boldsymbol{\rho}]_\times \mathrm{d}\mathbf{F}_{ext} \end{pmatrix} \right) \tag{3.14}$$

Please note that this formulation must hold for arbitrary virtual displacements as there are no active constraints from joints or contacts. Knowing the computation rule $\mathbf{a} \times (\mathbf{b} \times (\mathbf{b} \times \mathbf{a})) = -\mathbf{b} \times (\mathbf{a} \times (\mathbf{a} \times \mathbf{b}))$ and introducing

$$\int_{\mathcal{B}} \mathrm{d}m =: m \qquad \text{body mass} \tag{3.15}$$

$$\int_{\mathcal{B}} \boldsymbol{\rho} \mathrm{d}m =: \mathbf{0} \qquad \text{since S = COG} \tag{3.16}$$

$$\int_{\mathcal{B}} -[\boldsymbol{\rho}]_\times^2 \mathrm{d}m = \int_{\mathcal{B}} [\boldsymbol{\rho}]_\times [\boldsymbol{\rho}]_\times^T \mathrm{d}m =: \mathbf{\Theta}_S \qquad \text{Inertia matrix around COG} \tag{3.17}$$

we get

$$0 = \delta W = \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \mathbf{\Phi} \end{pmatrix}^T \left(\begin{bmatrix} \mathbb{I}_{3\times 3} m & \mathbf{0} \\ \mathbf{0} & \mathbf{\Theta}_S \end{bmatrix} \begin{pmatrix} \mathbf{a}_s \\ \mathbf{\Psi} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ [\mathbf{\Omega}]_\times \mathbf{\Theta}_S \mathbf{\Omega} \end{pmatrix} - \begin{pmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext} \end{pmatrix} \right) \quad \forall \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \mathbf{\Phi} \end{pmatrix}. \tag{3.18}$$

In order to define the laws of conservation of linear and angular momentum we introduce the definitions:

$$\mathbf{p}_S = m\mathbf{v}_S \qquad \text{linear momentum} \tag{3.19}$$

$$\mathbf{N}_S = \mathbf{\Theta}_S \cdot \mathbf{\Omega} \qquad \text{angular momentum around COG} \tag{3.20}$$

$$\dot{\mathbf{p}}_S = m\mathbf{a}_S \qquad \text{change in linear momentum} \tag{3.21}$$

$$\dot{\mathbf{N}}_S = \mathbf{\Theta}_S \cdot \mathbf{\Psi} + \mathbf{\Omega} \times \mathbf{\Theta}_S \cdot \mathbf{\Omega} \qquad \text{change in angular momentum} \tag{3.22}$$

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$$0 = \delta W = \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \mathbf{\Phi} \end{pmatrix}^T \left(\begin{pmatrix} \dot{\mathbf{p}}_S \\ \dot{\mathbf{N}}_S \end{pmatrix} - \begin{pmatrix} \mathbf{F}_{ext} \\ \mathbf{T}_{ext} \end{pmatrix} \right) \quad \forall \begin{pmatrix} \delta \mathbf{r}_s \\ \delta \mathbf{\Phi} \end{pmatrix}, \tag{3.23}$$

which results in the well-known formulations by Newton and Euler:

$$\dot{\mathbf{p}}_S = \mathbf{F}_{ext,S} \tag{3.24}$$

$$\dot{\mathbf{N}}_S = \mathbf{T}_{ext} \tag{3.25}$$

where $\mathbf{F}_{ext,S}$ are the resultant external forces that act through the COG and $\mathbf{T}_{ext}$ are the resultant external torques. External forces which do not act through the COG need to be shifted to an equivalent force/moment pair of which the force acts through the COG. Please note again that for numerical calculation, the terms of the change in linear and angular momentum must be expressed in the same coordinate system. For the inertia tensor $\mathbf{\Theta}$ we must apply ${}_{\mathcal{B}}\mathbf{\Theta} = \mathbf{C}_{\mathcal{B}\mathcal{A}} \cdot {}_{\mathcal{A}}\mathbf{\Theta} \cdot \mathbf{C}_{\mathcal{B}\mathcal{A}}^T$.