Poisson's and Laplace's Equations

For electrostatic field, we have seen that

$$\nabla \cdot \vec{D} = \rho_v$$
$$\vec{E} = -\nabla V \text{ ....} (2.97)$$

Form the above two equations we can write

$$\nabla \cdot (\varepsilon \vec{E}) = \nabla \cdot (-\varepsilon \nabla V) = \rho_v \text{ ....} (2.98)$$

Using vector identity we can write, $\varepsilon \nabla \cdot \nabla V + \nabla V \cdot \nabla \varepsilon = -\rho_v$ ....(2.99)

For a simple homogeneous medium, ε is constant and $\nabla \varepsilon = 0$. Therefore,

$$\nabla \cdot \nabla V = \nabla^2 V = -\frac{\rho_v}{\varepsilon} \text{ ....} (2.100)$$

This equation is known as **Poisson's equation**. Here we have introduced a new operator, ∇^2 (del square), called the Laplacian operator. In Cartesian coordinates,

$$\nabla^2 V = \nabla \cdot \nabla V = \left(\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right) \cdot \left(\frac{\partial V}{\partial x}\hat{a}_x + \frac{\partial V}{\partial y}\hat{a}_y + \frac{\partial V}{\partial z}\hat{a}_z\right) \text{ ....} (2.101)$$

Therefore, in Cartesian coordinates, Poisson equation can be written as:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x^2} = -\frac{\rho_v}{\varepsilon} \quad ....(2.102)$$

In cylindrical coordinates,

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \quad ....(2.103)$$

In spherical polar coordinate system,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \quad ....(2.104)$$

At points in simple media, where no free charge is present, Poisson's equation reduces to

$$\nabla^2 V = 0 \quad ....(2.105)$$

which is known as Laplace's equation.

Laplace's and Poisson's equation are very useful for solving many practical electrostatic field problems where only the electrostatic conditions (potential and charge) at some boundaries are known and solution of electric field and potential is to be found throughout the volume. We shall consider such applications in the section where we deal with boundary value problems.