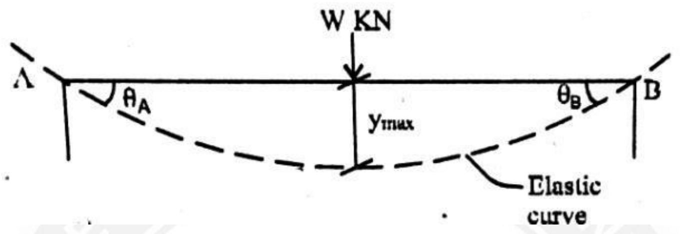
UNIT IV

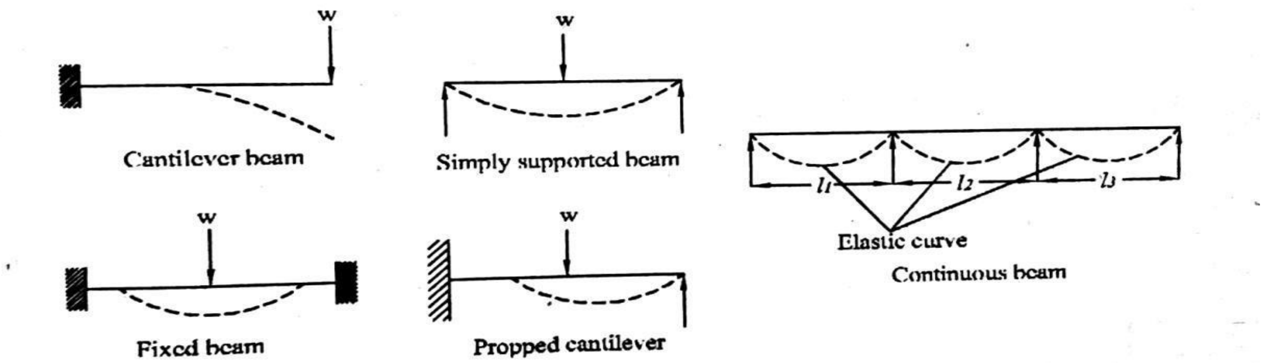
DEFLECTION OF BEAMS

4.1.ELASTIC CURVE OR DEFLECTED SHAPE

The curved shape of the longitudinal centroidal surface of a beam due to transverse loads is known as Elastic curve.



4.2.DEFLECTED SHAPES (or) ELASTIC CURVES OF BEAMS WITH DIFFERENT SUPPORT CONDITIONS



4.3.SLOPE

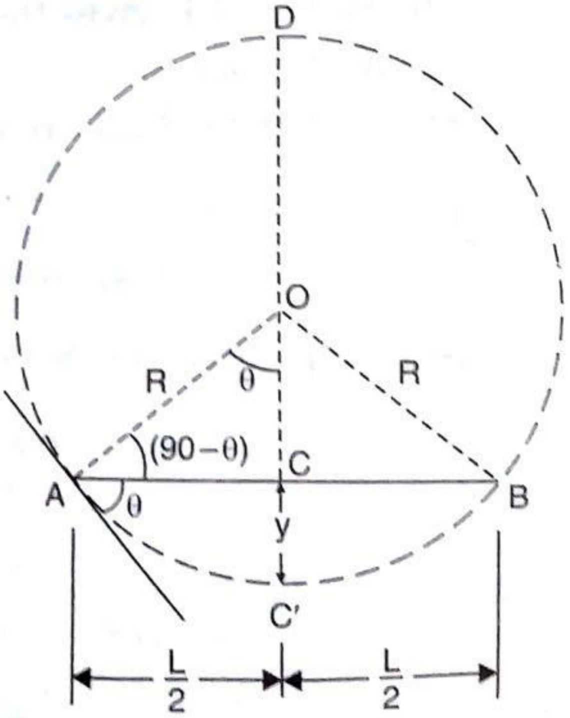
Slope is the angle formed by the tangent drawn at the Elastic curve to the original axis of the beam

4.4.DEFLECTION

Deflection is the translational movement of the beam from its original position.

4.5.DEFLECTION AND SLOPE OF A BEAM SUBJECTED TO UNIFORM BENDING MOMENT

A beam AB of length L is subjected to a uniform bending moment M as shown in Fig. As the beam is subjected to a constant bending moment, hence it will bend into a circular arc. The initial position of the beam is shown by ACB, whereas the deflected position is shown by AC'B.



Let R = Radius of curvature of the deflected beam, y = Deflection of the beam at the centre (i.e., distance CC') I = Moment of inertia of the beam section,

E = Young's modulus for the beam material, and

θ = Slope of the beam at the end A (i.e., the angle made by the tangent at A with the beam AB). For a practical beam the deflection y is a small Quantity.

Hence tan θ = θ where θ is in radians. Here θ becomes the slope

$\frac{dy}{dx} = \tan\theta = \theta$.

Now, $AC = BC = \frac{L}{2}$

Also from the geometry of a circle, we know that

AC X CB = DC X CC'

$\frac{L}{2} \times \frac{L}{2} = (2R - y) \times y$

$\frac{L^2}{4} = 2Ry - y^2$

For a practical beam, the deflection y is a small quantity. Hence the square of a small quantity will be negligible. Hence neglecting y^2 in the above equation, we get

$\frac{L^2}{4} = 2Ry$

$\therefore y = \frac{L^2}{8R}$ …(i)

But from bending equation, we have

$\frac{M}{I} = \frac{E}{R}$

Or $R = \frac{E \times I}{M}$ …(ii)

Substituting the value of R in equation (i), we get

$y = \frac{L^2}{8 \times \frac{EI}{M}}$

$y = \frac{M L^2}{8EI}$ …(iii)

Equation (iii) gives the central deflection of a beam which bends in a circular arc

Value of slope(θ)

From triangle AOB, we know that

$\sin\theta = \frac{AC}{AO} = \frac{\left[\frac{L}{2}\right]}{R} = \frac{L}{2R}$

since the angle θ is very small, hence sin θ = θ (in radians)

$\therefore \theta = \frac{L}{2R}$

$= \frac{L}{2 \times \frac{EI}{M}}$

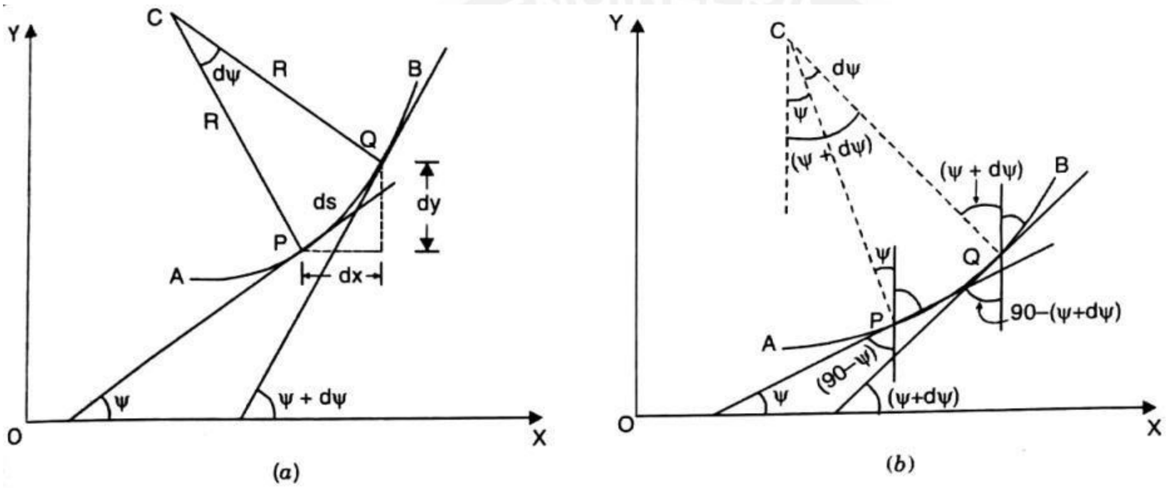
$= \frac{M \times L}{2EI}$ …(iv)

Equation (iv) gives the slope of the deflected beam at A or at B

4.6.RELATION BETWEEN SLOPE, DEFLECTION AND RADIUS OF CURVATURE or DERIVATION DIFFERENTIAL EQUATION

Let the curve AB represents the deflection of a beam as shown in Fig. Consider a small portion PQ of this beam. Let the tangents at P and Q make angle Ψ and Ψ + d Ψ with x axis.Normal at P and Q will meet at C such that

PC = QC = R



The point C is known as the centre of curvature of the curve PQ.

Let the length of PQ is equal to ds.

From fig.3.4.b we see that

Angle PCQ = d Ψ

PQ = ds = R.dΨ

$R = \frac{ds}{d\Psi}$ …(i)

But if x and y be the coordinates of P, then

$\tan \Psi = \frac{dy}{dx}$ …(ii)

$\sin \Psi = \frac{dy}{ds}$

and $\cos \Psi = \frac{dx}{ds}$

Now equatioin (i) can be written as

R = $\frac{ds}{d\Psi} = \frac{\left[\frac{ds}{dx}\right]}{\left(\frac{d\Psi}{dx}\right)} = \frac{\left[\frac{1}{cos\Psi}\right]}{\left(\frac{d\Psi}{dx}\right)}$

R = $\frac{sec\Psi}{\left(\frac{d\Psi}{dx}\right)}$ …(iii)

Differentiating equation (ii) w.r.t.x, we get

$$Sec^2\,\Psi \frac{d\Psi}{dx} = \frac{d^2y}{dx^2}$$

$$\frac{d\Psi}{dx} = \frac{\frac{d^2y}{dx^2}}{sec^2\Psi}$$

Substituting this value of $\frac{d\Psi}{dx}$ in equation (iii), we get

$$R = \frac{sec\Psi}{\left[\frac{\frac{d^2y}{dx^2}}{sec^2\Psi}\right]} = \frac{sec\Psi sec^2\Psi}{\frac{d^2y}{dx^2}} = \frac{sec^3\Psi}{\frac{d^2y}{dx^2}}$$

Taking the reciprocal to both sides, we get

$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{sec^3\Psi} = \frac{\frac{d^2y}{dx^2}}{(sec^2\Psi)^{3/2}}$$

$$= \frac{\frac{d^2y}{dx^2}}{(1+tan^2\Psi)^{3/2}}$$

For a practical beam, the slope tanΨ at any point is a small quantity. Hence $tan^2\Psi$ can be neglected.

$$\therefore \frac{1}{R} = \frac{d^2y}{dx^2} \quad \text{…(iv)}$$

From the bending equation, we have

$$\frac{M}{I} = \frac{E}{R}$$

Or $\frac{1}{R} = \frac{M}{EI}$ …(v)

Equating equations (iv) and (v), we get

$$\frac{M}{EI} = \frac{d^2y}{dx^2}$$

∴ $M = EI\frac{d^2y}{dx^2}$ …(vi)

Differentiating the above equation w.r.t.x, we get

$\frac{dM}{dx} = EI\frac{d^3y}{dx^3}$

But $\frac{dM}{dx}$ = F shear force

$\therefore F = EI\frac{d^3y}{dx^3}$…(vii)

Differentiating equation (vii) w.r.t.x., we get

$\frac{dF}{dx} = EI\frac{d^4y}{dx^4}$

But $\frac{dF}{dx}$= w the rate of loading

$\therefore w = EI\frac{d^4y}{dx^4}$

Hence, the relation between curvature, slope, deflection etc. at a section is given by

Deflection = y

Slope $= \frac{dy}{dx}$

Bending moment = $EI\frac{d^2y}{dx^2}$

Shear Force = $EI\frac{d^3y}{dx^3}$

The rate of loading = $EI\frac{d^4y}{dx^4}$