4.3 Z-TRANSFORM

The z-transform of a sequence x[n] is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}.$$

The z-transform can also be thought of as an operator Z{·} that transforms a sequence to a function:

$$\mathcal{Z}\{x[n]\} = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z).$$

The Fourier transform does not converge for all sequences—the infinite sum may not always be finite. Similarly, the z-transform does not converge for all sequences or for all values of z. The set of values of z for which the z-transform converges is called the **region of convergence (ROC)**.

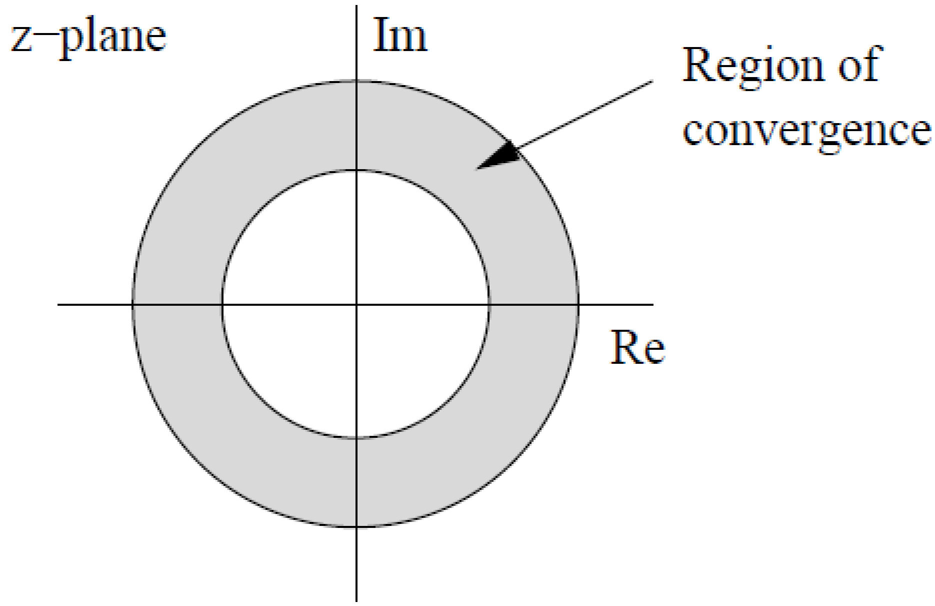
The z-transform therefore exists (or converges) if

$$X(z) = \sum_{n=-\infty}^{\infty} |x[n]r^{-n}| < \infty.$$

This leads to the condition

$$\sum_{n=-\infty}^{\infty} |x[n]||z|^{-n} < \infty$$

for the existence of the z-transform. The ROC therefore consists of a ring in the z-plane:



The inner radius of this ring may include the origin, and the outer radius may extend to infinity. If the ROC includes the unit circle |z| = 1,then the Fourier transform will converge.

PROPERTIES OF THE REGION OF CONVERGENCE

The properties of the ROC depend on the nature of the signal. Assuming that the signal has a finite amplitude and that the z-transform is a rational function:

• The ROC is a ring or disk in the z-plane, centered on the origin

$$(0 \leq r_R < |z| < r_L \leq \infty).$$

• The Fourier transform of x[n] converges absolutely if and only if the ROC of the z-transform includes the unit circle.

• The ROC cannot contain any poles.

• If x[n] is finite duration (ie. zero except on finite interval

$$-\infty < N_1 \leq n \leq N_2 < \infty),$$

then the ROC is the entire z-plane except perhaps at z = 0 or z = ∞.

• If x[n] is a right-sided sequence then the ROC extends outward from the outermost finite pole to infinity.

• If x[n] is left-sided then the ROC extends inward from the innermost nonzero pole to z = 0.

• A two-sided sequence (neither left nor right-sided) has a ROC consisting of a ring in the z-plane, bounded on the interior and exterior by a pole (and not containing any poles).

• The ROC is a connected region.

PROPERTIES OF Z-TRANSFORM

Linearity:

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \overset{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then linearity property states that

$$a\,x(n) + b\,y(n) \overset{\text{Z.T}}{\longleftrightarrow} a\,X(Z) + b\,Y(Z)$$

Time shifting:

The time-shifting property is as follows:

$$x[n-n_0] \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0} X(z), \qquad \text{ROC} = R_x.$$

Proof:

$$Y(z) = \sum_{n=-\infty}^{\infty} x[n-n_0] z^{-n} = \sum_{m=-\infty}^{\infty} x[m] z^{-(m+n_0)}$$

$$= z^{-n_0} \sum_{m=-\infty}^{\infty} x[m] z^{-m} = z^{-n_0} X(z).$$

Differentiation:

The differentiation property states that

$$nx[n] \xleftrightarrow{\mathcal{Z}} -z\frac{dX(z)}{dz}, \qquad \text{ROC} = R_x.$$

Proof:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n},$$

we have

$$-z\frac{dX(z)}{dz} = -z\sum_{n=-\infty}^{\infty}(-n)x[n]z^{-n-1} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n} = \mathcal{Z}\{nx[n]\}.$$

Conjugation:

$$x^*[n] \xleftrightarrow{\mathcal{Z}} X^*(z^*), \qquad \text{ROC} = R_x.$$

Convolution Property:

If $x(n) \xleftrightarrow{\text{Z.T}} X(Z)$

and

$$y(n) \xleftrightarrow{\text{Z.T}} Y(Z)$$

Then convolution property states that

$$x(n) * y(n) \xleftrightarrow{\text{Z.T}} X(Z).Y(Z)$$

Correlation Property:

If $x(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z)$

and $y(n) \overset{\text{Z.T}}{\longleftrightarrow} Y(Z)$

Then convolution property states that

$$x(n) * y(n) \overset{\text{Z.T}}{\longleftrightarrow} X(Z).Y(Z)$$

Initial Value and Final Value Theorems

Initial value and final value theorems of z-transform are defined for causal signal.

Initial Value Theorem

For a causal signal x(n) the initial value theorem states that

$$x(0) = \lim_{z\to\infty} X(z)$$

This is used to find the initial value of the signal without taking inverse z-transform

Final Value Theorem

For a causal signal x(n), the final value theorem states that

$$x(\infty) = \lim_{z\to 1}[z-1]X(z)$$

This is used to find the final value of the signal without taking inverse z-transform.