Curl of a vector field

We have defined the circulation of a vector field *A* around a closed path as $\oint_l \vec{A} \cdot d\vec{l}$.

Curl of a vector field is a measure of the vector field's tendency to rotate about a point. Curl $\vec{A}$, also written as $\nabla \times \vec{A}$

is defined as a vector whose magnitude is maximum of the net circulation per unit area when the area tends to zero and its direction is the normal direction to the area when the area is oriented in such a way so as to make the circulation maximum.

Therefore, we can write:

$$Curl\ \vec{A} = \nabla \times \vec{A} = \lim_{\Delta S \to 0} \frac{\hat{a}_n}{\Delta S} \left[\oint_l \vec{A} \cdot dl \right]_{max} \text{......................................(1.68)}$$

To derive the expression for curl in generalized curvilinear coordinate system, we first compute $\nabla \times \vec{A} \cdot \hat{a}_u$ and to do so let us consider the figure 1.20 :

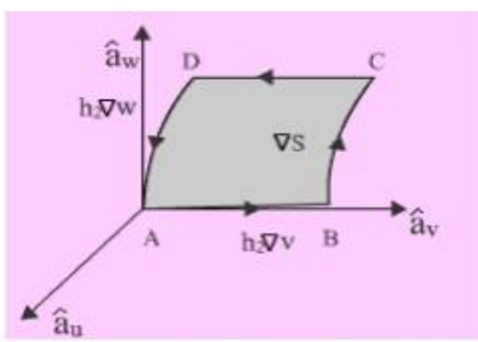

Fig 1.20: Curl of a Vector

C_1 represents the boundary of ΔS, then we can write

$$\oint_{C_1} \vec{A}\cdot d\vec{l} = \int_{AB} \vec{A}\cdot d\vec{l} + \int_{BC} \vec{A}\cdot d\vec{l} + \int_{CD} \vec{A}\cdot d\vec{l} + \int_{DA} \vec{A}\cdot d\vec{l} \quad \text{.......................................(1.69)}$$

The integrals on the RHS can be evaluated as follows:

$$\int_{AB} \vec{A}\cdot d\vec{l} = \left(A_u\hat{a}_u + A_v\hat{a}_v + A_w\hat{a}_w\right)\cdot h_2\Delta v\hat{a}_v = A_v h_2 \Delta v \quad \text{..................................(1.70)}$$

$$\int_{CD} \vec{A}\cdot d\vec{l} = -\left(A_v h_2 \Delta v + \frac{\partial}{\partial w}(A_v h_2 \Delta v)\Delta w\right) \quad \text{..(1.71)}$$

The negative sign is because of the fact that the direction of traversal reverses. Similarly,

$$\int_{BC} \vec{A}\cdot d\vec{l} = \left(A_w h_3 \Delta w + \frac{\partial}{\partial v}(A_w h_3 \Delta w)\Delta v\right) \quad \text{..(1.72)}$$

$$\int_{DA} \vec{A}\cdot d\vec{l} = -A_w h_3 \Delta w \quad \text{...(1.73)}$$

Adding the contribution from all components, we can write:

$$\oint_{C_1} \vec{A}\cdot d\vec{l} = \left(\frac{\partial}{\partial v}(A_w h_3) - \frac{\partial}{\partial w}(A_v h_3)\right)\Delta v \Delta w \quad \text{...(1.74)}$$

Therefore, $$\left(\nabla\times\vec{A}\right).\hat{a}_u = \frac{\oint_{C_1} \vec{A}\cdot d\vec{l}}{h_2 h_3 \Delta v \Delta w} = \frac{1}{h_2 h_3}\left(\frac{\partial(h_3 A_w)}{\partial v} - \frac{\partial(h_2 A_v)}{\partial w}\right) \quad \text{........................(1.75)}$$

In the same manner if we compute for $(\nabla\times\vec{A}).\hat{a_v}$ and $(\nabla\times\vec{A}).\hat{a_w}$ we can write,

$$\nabla\times\vec{A} = \frac{1}{h_2h_3}\left(\frac{\partial(h_3A_w)}{\partial v}-\frac{\partial(h_2A_v)}{\partial w}\right)\hat{a}_u + \frac{1}{h_1h_3}\left(\frac{\partial(h_1A_u)}{\partial w}-\frac{\partial(h_3A_w)}{\partial u}\right)\hat{a}_v + \frac{1}{h_1h_2}\left(\frac{\partial(h_2A_v)}{\partial u}-\frac{\partial(h_1A_u)}{\partial v}\right)\hat{a}_w$$
.......(1.76)

This can be written as,

$$\nabla\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{a}_u & h_2\hat{a}_v & h_3\hat{a}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_1A_u & h_2A_v & h_3A_w \end{vmatrix}$$
....(1.77)

In Cartesian coordinates:

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$
....(1.78)

In Cylindrical coordinates,

$$\nabla\times\vec{A} = \frac{1}{\rho}\begin{vmatrix} \hat{a}_\rho & \rho\hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & \rho A_\phi & A_z \end{vmatrix}$$
.....................................(1.79)

In Spherical polar coordinates,

$$\nabla\times\vec{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta\hat{a}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ A_r & rA_\theta & r\sin\theta A_\phi \end{vmatrix} \quad ....(1.80)$$

Curl operation exhibits the following properties:

(i) *Curl of a vector field is another vector field.*

(ii) $\nabla\times(\vec{A}+\vec{B}) = \nabla\times\vec{A}+\nabla\times\vec{B}$

(iii) $\nabla\times(V\vec{A}) = \nabla V\times\vec{A}+V\nabla\times\vec{A}$

(iv) $\nabla.(\nabla\times\vec{A}) = 0$

(v) $\nabla\times\nabla V = 0$

(vi) $\nabla\times(\vec{A}\times\vec{B}) = \vec{A}\nabla.\vec{B}-\vec{B}\nabla.\vec{A}+(\vec{B}.\nabla)\vec{A}-(\vec{A}.\nabla)\vec{B}$ ....(1.81)

Stoke's theorem :

It states that the circulation of a vector field $\vec{A}$ around a closed path is equal to the integral of $\nabla\times\vec{A}$ over the surface bounded by this path. It may be noted that this equality holds provided $\vec{A}$ and $\nabla\times\vec{A}$ are continuous on the surface.

i.e,

$$\oint_L \vec{A}\cdot d\vec{l} = \int_s \nabla\times\vec{A}\cdot d\vec{s} \quad ....(1.82)$$

Proof: Let us consider an area S that is subdivided into large number of cells as shown in the figure 1.21.

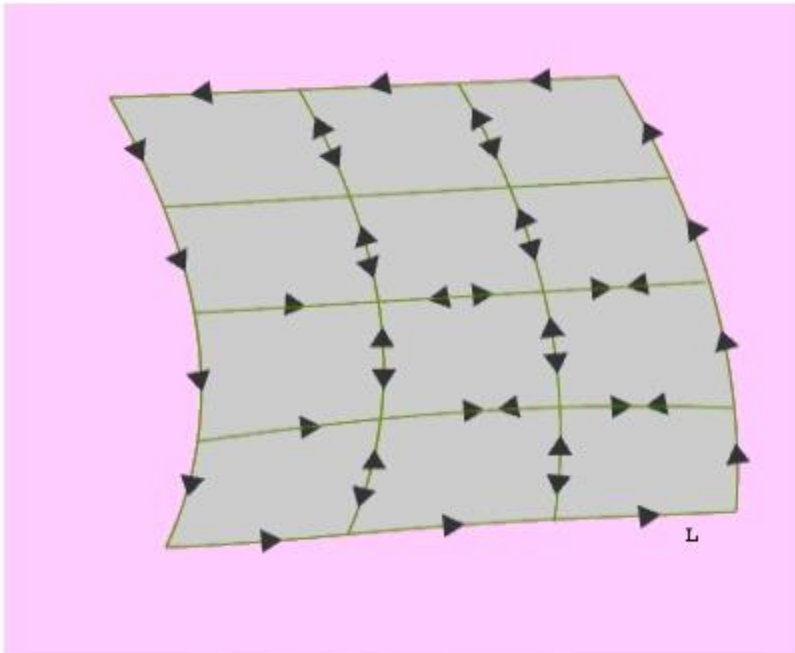

Fig 1.21: Stokes theorem

Let kth cell has surface area □Sk and is bounded path Lk while the total area is bounded by path L. As seen from the figure that if we evaluate the sum of the line integrals around the elementary areas, there is cancellation along every interior path and we are left the line integral along path L. Therefore we can write,

$$\oint_L \vec{A}\cdot d\vec{l} = \sum_k \oint_{L_k} \vec{A}\cdot d\vec{l} = \sum_k \frac{\oint_{L_k} \vec{A}\cdot d\vec{l}}{\Delta S_k}\Delta S_k \quad \text{..............(1.83)}$$

As $\Delta S_k \rightarrow 0$

$$\oint_L \vec{A}\cdot d\vec{l} = \int_S \nabla\times\vec{A}\cdot d\vec{s} \quad \text{.............(1.84)}$$

which is the stoke's theorem.