UNIT III APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

3.1 ONE DIMENSIONAL WAVE EQUATION

One Dimensional Wave Equation: $\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$

↓

Vibrations of a string

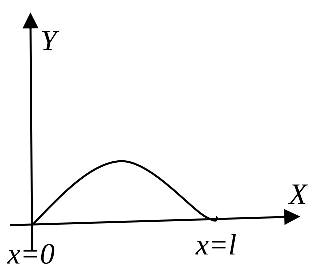
↓

Zero Initial velocity

Hint:
1) Velocity will not be given in the question
2) The word released from the rest will be there in the question

↓

The Boundary and Initial conditions are:
i) $y(0,t) = 0$
ii) $y(l,t) = 0$
iii) $\frac{\partial y}{\partial t}(x,0) = 0$
iv) $y(x,0) = f(x),\ 0 \le x \le l$



Non Zero Initial velocity

Hint:
1) Velocity will be given in the question
2) The word initially at rest will be there in the question

↓

The Boundary and Initial conditions are:
i) $y(0,t) = 0$
ii) $y(l,t) = 0$
iii) $y(x,0) = 0$
iv) $\frac{\partial y}{\partial t}(x,0) = f(x),\ 0 \le x \le l$

↓

The Correct (or) Suitable Solution is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$$

$y(x,t) \rightarrow$ The dis

1) **In the wave equation** $\frac{\partial^2 y}{\partial t^2} = C^2 \frac{\partial^2 y}{\partial x^2}$**, what does** C^2 **stands for?**

Solution:

One dimensional heat equation is $\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2}$

$C^2 = T/m$, where T is the tension and m is the mass of the string.

2) **Write all possible solutions of the transverse vibration of the string in one dimension.**

Solution:

(i) $y(x,t) = \left(Ae^{px} + Be^{-px}\right)\left(Ce^{pat} + De^{-pat}\right)$

(ii) $y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat)$.

(iii) $y(x,t) = (Ax + B)(Ct + D$

PART - B

1. **A uniform string is stretched and fastened to two points $'l'$ apart. Motion is started by displacing the string into the form of the curve $y = kx(l-x)$ and then releasing it from this position at time $t = 0$. Find the displacement of the point of the string at a distance x from one end at time t.**

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$

The correct solution is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \quad -----(1)$$

The Boundary and Initial conditions are

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $\frac{\partial y}{\partial t}(x,0) = 0$

iv) $y(x,0) = f(x) = kx(l-x), \quad 0 < x < l$

Applying condn (i) in (1)

$$(1) \Rightarrow y(0,t) = (A\cos 0 + \cancel{B\sin 0})(C\cos pat + D\sin pat)$$

$$0 = (A)(C\cos pat + D\sin pat)$$

Here $(C\cos pat + D\sin pat) \neq 0 \quad \therefore \boxed{A = 0}$

Sub A=0 in (1)

$$(1) \Rightarrow y(x,t) = (B\sin px)(C\cos pat + D\sin pat) \quad -----(2)$$

Applying condn (ii) in (2)

$$y(l,t)=(B\sin pl)(C\cos pat + D\sin pat)$$

$$0=(B\sin pl)(C\cos pat + D\sin pat)$$

Here $B \neq 0,\ (C\cos pat + D\sin pat)\neq 0$

$$\therefore \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p=\frac{n\pi}{l}}$$

Sub the value of p in (2)

$$(2)\Rightarrow\ y(x,t)=\left(B\sin\frac{n\pi}{l}x\right)\left(C\cos\frac{n\pi a}{l}t + D\sin\frac{n\pi a}{l}t\right)\ \ ------(3)$$

Diff (3) partially w.r.to 't'

$$(2)\Rightarrow\ \frac{\partial y}{\partial t}(x,t)=\left(B\sin\frac{n\pi}{l}x\right)\left[-C\sin\frac{n\pi a}{l}t\times\left(\frac{n\pi a}{l}\right)+D\cos\frac{n\pi a}{l}t\times\left(\frac{n\pi a}{l}\right)\right]$$

Apply condn. (iii) in the above equation

$$\frac{\partial y}{\partial t}(x,0)=\left(B\sin\frac{n\pi}{l}x\right)\left[\cancel{-C\sin 0\times\left(\frac{n\pi a}{l}\right)}+D\cos 0\times\left(\frac{n\pi a}{l}\right)\right]$$

$$0=\left(B\sin\frac{n\pi}{l}x\right)\left[D\times\left(\frac{n\pi a}{l}\right)\right]$$

Here $B\neq 0$, $\sin\frac{n\pi}{l}x\neq 0,\ \frac{n\pi a}{l}\neq 0,\ \therefore \boxed{D=0}$

Sub the value of D in (3)

$$(3)\Rightarrow\ y(x,t)=\left(B\sin\frac{n\pi}{l}x\right)\left(C\cos\frac{n\pi a}{l}t+0\right)$$

$$y(x,t)=BC\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t$$

$$y(x,t) = b_1 \sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t \qquad \text{let } BD = b_1$$

The most general solution is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t} ------(4)$$

Applying condn (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\cos 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \qquad \because \cos 0 = 1$$

Which is half range Fourier sine series in $(0,l)$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx$$

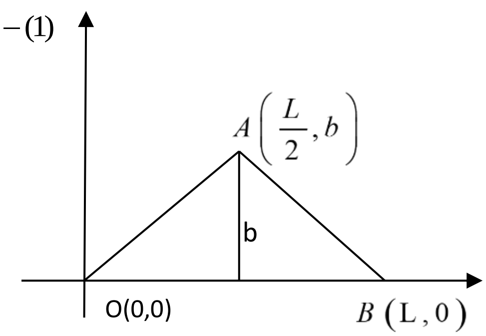
$$= \frac{2}{l}\int_0^l kx(l-x)\sin\frac{n\pi x}{l}dx$$

$$= \frac{2k}{l}\left[\left(lx - x^2\right)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \left(l-2x\right)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + \left(-2\right)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0^l$$

$$= \frac{2k}{l}\left[\frac{-2l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{-4kl^3}{ln^3\pi^3}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{-4kl^2}{n^3\pi^3}\left[(-1)^n - 1\right]$$

	$b_n = \begin{cases} 0 & if\ n\ is\ even \\ \frac{8kl^2}{n^3\pi^3} & if\ n\ is\ odd \end{cases}$ Sub b_n in (4) $y(x,t) = \sum_{n=1,3,5,...}^{\infty} \frac{8kl^2}{n^3\pi^3} \sin\frac{n\pi}{l}x \cos\frac{n\pi a}{l}t$ (or) $y(x,t) = \frac{8kl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{(2n-1)\pi}{l}x\cos\frac{(2n-1)\pi a}{l}t$
2.	**A string of length *2l* is fastened at both ends. The midpoint of the string is displaced transversely through a small distance *'b'* and the string is released from the rest in that position. Find an expression for the transverse displacement of the string at any time during the subsequent motion.** **Solution:** One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$ The correct solution is $y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \quad -----(1)$  The Boundary and Initial conditions are Assume *2l=L* i) $y(0,t) = 0$ ii) $y(\mathrm{L},t) = 0$ iii) $\frac{\partial y}{\partial t}(x,0) = 0$ iv) $y(x,0) = f(x) = ?$ **To find *f(x)*:** **The equation of line joining two points is**

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}$$

Eqution of OA is ***O(0,0)& A(l/2,b)*****:**

$$\frac{x-0}{\frac{L}{2}-0}=\frac{y-0}{b-0}\Rightarrow \frac{2x}{L}=\frac{y}{b}\Rightarrow y=\frac{2b}{L}x,\ 0<x<\frac{L}{2}$$

Eqution of AB is ***A(L/2,b)&B(L,0):***

$$\frac{x-\frac{L}{2}}{L-\frac{L}{2}}=\frac{y-b}{0-b}\Rightarrow \frac{\frac{2x-L}{2}}{\frac{L}{2}}=\frac{y-b}{-b}\Rightarrow \frac{2x-L}{L}=\frac{y-b}{-b}$$

$$-2xb+lb=yl-lb$$

$$-2xb+Lb+Lb=yL\Rightarrow -2xb+2Lb=yL$$

$$y=\frac{2b}{L}(\mathrm{L}-x),\ \frac{L}{2}<x<L$$

$$y=f(x)=\begin{cases}\frac{2b}{L}x, & 0<x<\frac{L}{2}\\ \frac{2b}{L}(\mathrm{L}-x), & \frac{L}{2}<x<L\end{cases}$$

Applying condn (i) in (1)

$$(1)\Rightarrow y(0,t)=(A\cos 0+\cancel{B\sin 0})(C\cos pat+D\sin pat)$$

$$0=(A)(C\cos pat+D\sin pat)$$

Here $(C\cos pat+D\sin pat)\neq 0\ \therefore \boxed{\mathrm{A}=0}$

Sub A=0 in (1)

$$(1)\Rightarrow y(x,t)=(B\sin px)(C\cos pat+D\sin pat)\ -----(2)$$

Applying condn (ii) in (2)

$y(\mathrm{L},t)=(B\sin pL)(C\cos pat+D\sin pat)$

$0=(B\sin pL)(C\cos pat+D\sin pat)$

Here $\mathrm{B}\neq 0,\ (C\cos pat+D\sin pat)\neq 0$

$\therefore \sin pL=0 \Rightarrow \sin pL=\sin n\pi \Rightarrow pL=n\pi \Rightarrow \boxed{p=\frac{n\pi}{L}}$

Sub the value of p in (2)

$$(2)\Rightarrow\ y(x,t)=\left(B\sin\frac{n\pi}{L}x\right)\left(C\cos\frac{n\pi a}{L}t+D\sin\frac{n\pi a}{L}t\right)\ \text{-----}(3)$$

Diff (3) partially w.r.to 't'

$$(2)\Rightarrow\ \frac{\partial y}{\partial t}(x,t)=\left(B\sin\frac{n\pi}{L}x\right)\left[-C\sin\frac{n\pi a}{L}t\times\left(\frac{n\pi a}{L}\right)+D\cos\frac{n\pi a}{L}t\times\left(\frac{n\pi a}{L}\right)\right]$$

Apply condn. (iii) in the above equation

$$\frac{\partial y}{\partial t}(x,0)=\left(B\sin\frac{n\pi}{L}x\right)\left[\cancel{-C\sin 0\times\left(\frac{n\pi a}{L}\right)}+D\cos 0\times\left(\frac{n\pi a}{L}\right)\right]$$

$$0=\left(B\sin\frac{n\pi}{L}x\right)\left[D\times\left(\frac{n\pi a}{L}\right)\right]$$

Here $B\neq 0$, $\sin\frac{n\pi}{L}x\neq 0,\ \frac{n\pi a}{L}\neq 0,\ \therefore D=0$

Sub the value of D in (3)

$$(3)\Rightarrow\ y(x,t)=\left(B\sin\frac{n\pi}{L}x\right)\left(C\cos\frac{n\pi a}{L}t+0\right)$$

$$y(x,t)=BC\sin\frac{n\pi}{L}x\cos\frac{n\pi a}{L}t$$

$$y(x,t)=b_1\sin\frac{n\pi}{L}x\cos\frac{n\pi a}{L}t \qquad \text{let } BD=b_1$$

The most general solution is

$$\boxed{y(x,t)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x\cos\frac{n\pi a}{L}t} -----(4)$$

Applying condn (iv) in (4)

$$y(x,0)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x\cos 0$$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{L}x \quad \because \cos 0=1$$

Which is half range Fourier sine series in $(0,l)$

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$=\frac{2}{L}\left[\int_0^{\frac{L}{2}}\frac{2b}{L}x\sin\frac{n\pi x}{L}dx+\int_{\frac{L}{2}}^{L}\frac{2b}{L}(L-x)\sin\frac{n\pi x}{L}dx\right] \quad \because y=f(x)=\begin{cases}\frac{2b}{L}x, & 0<x<\frac{L}{2}\\ \frac{2b}{L}(\mathrm{L}-x), & \frac{L}{2}<x<L\end{cases}$$

$$=\frac{4b}{L^2}\left[\int_0^{\frac{L}{2}}x\sin\frac{n\pi x}{L}dx+\int_{\frac{L}{2}}^{L}(L-x)\sin\frac{n\pi x}{L}dx\right]$$

$$=\frac{4b}{L^2}\left\{\left[(x)\left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right)-(1)\left(\frac{-\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right]_0^{\frac{L}{2}}+\left[(L-x)\left(\frac{-\cos\frac{n\pi x}{L}}{\frac{n\pi}{L}}\right)-(-1)\left(\frac{-\sin\frac{n\pi x}{L}}{\frac{n^2\pi^2}{L^2}}\right)\right]_{\frac{L}{2}}^{L}\right\}$$

$$=\frac{4b}{L^2}\left\{\left[-\frac{L}{n\pi}x\cos\frac{n\pi x}{L}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right]_0^{\frac{L}{2}}+\left[-\frac{L}{n\pi}(L-x)\cos\frac{n\pi x}{L}-\frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right]_{\frac{L}{2}}^{L}\right\}$$

$$=\frac{4b}{L^2}\left\{\left[\left(-\frac{L}{n\pi}\frac{L}{2}\cos\frac{n\pi}{2}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)-(0)\right]_0^{\frac{L}{2}}+\left[(0)-\left(-\frac{L}{n\pi}\left(\frac{L}{2}\right)\cos\frac{n\pi}{2}-\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]_{\frac{L}{2}}^{L}\right\}$$

$$=\frac{4b}{L^2}\left[\cancel{-\frac{L^2}{2n\pi}\cos\frac{n\pi}{2}}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\cancel{\frac{L^2}{2n\pi}\cos\frac{n\pi}{2}}+\frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$=\frac{4b}{L^2}\left[\frac{2L^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$\boxed{b_n=\frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}}$$

Sub b_n in (4)

$$\therefore y(x,t)=\sum_{n=1}^{\infty}\frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\cos\frac{n\pi at}{L}$$

$$\therefore y(x,t)=\frac{8b}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2l}\cos\frac{n\pi at}{2l}$$

This is the required displacement.

3. **A string of length *l* is fastened at both ends. The midpoint of the string is displaced transversely through a small distance '*b*' and the string is released from the rest in that position. Find an expression for the transeverse displacement of the string at any time during the subsequent motion.**

Solution:

Replace *L by l* in the above problem

4. **A tightly stretched string with fixed end points *x=0* and *x=l* is initially in a position given by** $y(x,0)=y_0\sin^3\frac{\pi x}{l}$ **. If it is released from rest from this position, find the displacement *y* at any distance *x* from one end any time *t.***

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$ where $a^2=\frac{T}{m}$

The correct solution is

$$y(x,t)=(A\cos px+B\sin px)(C\cos pat+D\sin pat)\quad -----(1)$$

The Boundary and Initial conditions are

i) $y(0,t)=0$

ii) $y(l,t)=0$

iii) $\frac{\partial y}{\partial t}(x,0)=0$

iv) $y(x,0)=f(x)=y_0\sin^3\frac{\pi x}{l}, \quad 0<x<l$

Applying condn (i) in (1)

$$(1)\Rightarrow y(0,t)=(A\cos 0+\cancel{B\sin 0})(C\cos pat+D\sin pat)$$

$$0=(A)(C\cos pat+D\sin pat)$$

Here $(C\cos pat+D\sin pat)\neq 0 \;\therefore \boxed{A=0}$

Sub A=0 in (1)

$$(1)\Rightarrow y(x,t)=(B\sin px)(C\cos pat+D\sin pat) \quad -----(2)$$

Applying condn (ii) in (2)

$$y(l,t)=(B\sin pl)(C\cos pat+D\sin pat)$$

$$0=(B\sin pl)(C\cos pat+D\sin pat)$$

Here $B\neq 0, \; (C\cos pat+D\sin pat)\neq 0$

$$\therefore \sin pl=0 \Rightarrow \sin pl=\sin n\pi \Rightarrow pl=n\pi \Rightarrow \boxed{p=\frac{n\pi}{l}}$$

Sub the value of p in (2)

$$(2)\Rightarrow y(x,t)=\left(B\sin\frac{n\pi}{l}x\right)\left(C\cos\frac{n\pi a}{l}t+D\sin\frac{n\pi a}{l}t\right) \quad -----(3)$$

Diff (3) partially w.r.to 't'

$$(2) \Rightarrow \frac{\partial y}{\partial t}(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left[-C\sin\frac{n\pi a}{l}t \times \left(\frac{n\pi a}{l}\right) + D\cos\frac{n\pi a}{l}t \times \left(\frac{n\pi a}{l}\right)\right]$$

Apply condn. (iii) in the above equation

$$\frac{\partial y}{\partial t}(x,0) = \left(B\sin\frac{n\pi}{l}x\right)\left[\cancel{-C\sin 0 \times \left(\frac{n\pi a}{l}\right)} + D\cos 0 \times \left(\frac{n\pi a}{l}\right)\right]$$

$$0 = \left(B\sin\frac{n\pi}{l}x\right)\left[D \times \left(\frac{n\pi a}{l}\right)\right]$$

Here $B \neq 0$, $\sin\frac{n\pi}{l}x \neq 0$, $\frac{n\pi a}{l} \neq 0$, $\therefore \boxed{D = 0}$

Sub the value of D in (3)

$$(3) \Rightarrow y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(C\cos\frac{n\pi a}{l}t + 0\right)$$

$$y(x,t) = BC\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t$$

$$y(x,t) = b_1\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t \qquad \text{let } BD = b_1$$

The most general solution is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t} \text{ - - - - - - } (4)$$

Applying condn (iv) in (4)

$$y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\cos 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \qquad \because \cos 0 = 1$$

	$y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$ $y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \qquad \because \sin^3\theta = \frac{1}{4}\left[3\sin\theta - \sin 3\theta\right]$ $y_0 \left\{\frac{1}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]\right\} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x$ $\frac{3y_0}{4}\sin\frac{\pi x}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l} = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3\sin\frac{3\pi x}{l} + b_4 \sin\frac{4\pi x}{l} + \dots$ Equating co-efficients of likely terms on both sides $b_1 = \frac{3y_0}{4}\,; b_2 = 0;\; b_3 - \frac{y_0}{4};\, b_4 = b_5 = b_6 \dots = 0.$ Sub these values in (4) $(4) \Rightarrow y(x,t) = b_1 \sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} + b_2 \sin\frac{2\pi x}{l}\cos\frac{2\pi a t}{l} + b_3 \sin\frac{3\pi x}{l}\cos\frac{3\pi a t}{l} + \dots.$ $\boxed{y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi a t}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi a t}{l}}$
5.	**A tightly stretched string end points *x=0* and *x=l* is initially at restin its equilibrium position. If it is set vibrating giving each point a velocity $\lambda(lx - x^2)$, then show that the displacement of given string is** $y(x,t) = \frac{8\lambda l^3}{a\pi^4}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi x}{l}\sin\frac{(2n-1)\pi a t}{l}$ **.** **Solution:** One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$ The correct solution is $y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \quad -----(1)$ The Boundary and Initial conditions are

	i) $y(0,t)=0$ ii) $y(l,t)=0$ iii) $y(x,0)=0$ iv) $\frac{\partial y}{\partial t}(x,0)=\lambda(lx-x^2),\ 0<x<l$ Applying condn (i) in (1) $(1)\Rightarrow\ y(0,t)=(A\cos 0+B\sin 0)(C\cos pat+D\sin pat)$ $0=(A)(C\cos pat+D\sin pat)$ Here $(C\cos pat+D\sin pat)\neq 0\ \therefore \boxed{A=0}$ Sub A=0 in (1) $(1)\Rightarrow\ y(x,t)=(B\sin px)(C\cos pat+D\sin pat)\ \ -----(2)$ Applying condn (ii) in (2) $y(l,t)=(B\sin pl)(C\cos pat+D\sin pat)$ $0=(B\sin pl)(C\cos pat+D\sin pat)$ Here $B\neq 0,\ (C\cos pat+D\sin pat)\neq 0$ $\therefore \sin pl=0\ \Rightarrow \sin pl=\sin n\pi\ \Rightarrow pl=n\pi\ \Rightarrow \boxed{p=\frac{n\pi}{l}}$ Sub the value of p in (2) $(2)\Rightarrow\ y(x,t)=\left(B\sin\frac{n\pi}{l}x\right)\left(C\cos\frac{n\pi a}{l}t+D\sin\frac{n\pi a}{l}t\right)\ \ -----(3)$ Apply condn. (iii) in the above equation

$$(3) \Rightarrow\ y(x,0) = \left(B\sin\frac{n\pi}{l}x\right)\left(C\cos 0 + \cancel{D\sin 0}\right)$$

$$0 = \left(B\sin\frac{n\pi}{l}x\right)(C)$$

Here $B \neq 0$, $\sin\frac{n\pi}{l}x \neq 0$, $\boxed{\therefore C = 0}$

Sub the value of C in (3)

$$(3) \Rightarrow\ y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(D\sin\frac{n\pi a}{l}t\right)$$

$$y(x,t) = BD\sin\frac{n\pi}{l}x\sin\frac{n\pi a}{l}t$$

$$y(x,t) = b_1\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t \qquad \text{let } BD = b_1$$

The most general solution is

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\sin\frac{n\pi a}{l}t} ------(4)$$

Diff (4) partially w.r.to 't'

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\ \cos\frac{n\pi a}{l}t \times \left(\frac{n\pi a}{l}\right)$$

Applying condn (iv) in the above eqn.

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x\ \cos 0 \times \left(\frac{n\pi a}{l}\right)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \times \left(\frac{n\pi a}{l}\right) \qquad \because \cos 0 = 1$$

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \qquad \text{let } B_n = b_n\frac{n\pi a}{l}$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{l} x$$

Which is half range Fourier sine series in $(0, l)$

$$B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2}{l} \int_0^l \lambda x(l - x) \sin \frac{n\pi x}{l} dx$$

$$= \frac{2\lambda}{l} \left[(lx - x^2) \left(\frac{-\cos \frac{n\pi x}{l}}{\frac{n\pi}{l}} \right) - (l - 2x) \left(\frac{-\sin \frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} \right) + (-2) \left(\frac{\cos \frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}} \right) \right]_0^l$$

$$= \frac{2\lambda}{l} \left[\frac{-2l^3}{n^3\pi^3} \cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{-4\lambda l^3}{l n^3 \pi^3} [\cos n\pi - \cos 0]$$

$$b_n \frac{n\pi a}{l} = \frac{-4\lambda l^2}{n^3\pi^3} \left[(-1)^n - 1 \right]$$

$$b_n = \frac{-4\lambda l^3}{n^4 \pi^4 a} \left[(-1)^n - 1 \right]$$

$$b_n = \begin{cases} 0 & if\ n\ is\ even \\ \dfrac{8\lambda l^3}{n^4\pi^4 a} & if\ n\ is\ odd \end{cases}$$

Sub b_n in (4)

$$y(x,t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{8\lambda l^3}{n^4\pi^4 a} \sin \frac{n\pi}{l} x \sin \frac{n\pi a}{l} t$$

	(or) $y(x,t)=\frac{8\lambda l^3}{\pi^4 a}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi}{l}x\sin\frac{(2n-1)\pi a}{l}t$
6.	**If a string of length l is initially at rest in its equilibrium position whose ends are fixed and each of its points is given a velocity v such that** $v=\begin{cases} cx;\ 0<x<\frac{l}{2} \\ c(l-x);\ \frac{l}{2}<x<l \end{cases}$ **, find the displacement of the string at any time t.** **Solution:** One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2}$ where $a^2=\frac{T}{m}$ The correct solution is $y(x,t)=(A\cos px+B\sin px)(C\cos pat+D\sin pat)\quad -----(1)$ The Boundary and Initial conditions are i) $y(0,t)=0$ ii) $y(l,t)=0$ iii) $y(x,0)=0$ iv) $\frac{\partial y}{\partial t}(x,0)=\begin{cases} cx;\ 0<x<\frac{l}{2} \\ c(l-x);\ \frac{l}{2}<x<l \end{cases}$ Applying condn (i) in (1) $(1)\Rightarrow y(0,t)=(A\cos 0+\cancel{B\sin 0})(C\cos pat+D\sin pat)$ $0=(A)(C\cos pat+D\sin pat)$ Here $(C\cos pat+D\sin pat)\neq 0$ $\boxed{\therefore A=0}$ Sub A=0 in (1)

$$(1) \Rightarrow \; y(x,t) = (B\sin px)(C\cos pat + D\sin pat) \quad ------(2)$$

Applying condn (ii) in (2)

$$y(l,t) = (B\sin pl)(C\cos pat + D\sin pat)$$

$$0 = (B\sin pl)(C\cos pat + D\sin pat)$$

Here $\mathrm{B} \neq 0, \; (C\cos pat + D\sin pat) \neq 0$

$$\therefore \sin pl = 0 \;\Rightarrow \sin pl = \sin n\pi \;\Rightarrow pl = n\pi \;\Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub the value of p in (2)

$$(2) \Rightarrow \; y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(C\cos\frac{n\pi a}{l}t + D\sin\frac{n\pi a}{l}t\right) \quad ------(3)$$

Apply condn. (iii) in the above equation

$$(3) \Rightarrow \; y(x,0) = \left(B\sin\frac{n\pi}{l}x\right)\left(C\cos 0 + \cancel{D\sin 0}\right)$$

$$0 = \left(B\sin\frac{n\pi}{l}x\right)(C)$$

Here $B \neq 0$, $\sin\frac{n\pi}{l}x \neq 0, \; \therefore C = 0$

Sub the value of C in (3)

$$(3) \Rightarrow \; y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(D\sin\frac{n\pi a}{l}t\right)$$

$$y(x,t) = BD\sin\frac{n\pi}{l}x\sin\frac{n\pi a}{l}t$$

$$y(x,t) = b_1\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t \qquad \text{let } BD = b_1$$

The most general solution is

$$\boxed{y(x,t)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \sin\frac{n\pi a}{l}t} ------(4)$$

Diff (4) partially w.r.to 't'

$$\frac{\partial y}{\partial t}(x,t)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \cos\frac{n\pi a}{l}t \times\left(\frac{n\pi a}{l}\right)$$

Applying condn (iv) in the above eqn.

$$\frac{\partial y}{\partial t}(x,0)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{l}x \cos 0 \times\left(\frac{n\pi a}{l}\right)$$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\times\left(\frac{n\pi a}{l}\right) \quad \because \cos 0 = 1$$

$$y(x,0)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{l} \quad \text{let } B_n = b_n\frac{n\pi a}{l}$$

$$f(x)=\sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{l}x$$

Which is half range Fourier sine series in $(0,l)$

$$B_n=\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

$$=\frac{2}{l}\left[\int_0^{\frac{l}{2}} cx\sin\frac{n\pi x}{l}\,dx+\int_{\frac{l}{2}}^{l} c(l-x)\sin\frac{n\pi x}{l}\,dx\right] \quad \because f(x)=\begin{cases} cx;\ 0<x<\frac{l}{2} \\ c(l-x);\ \frac{l}{2}<x<l\end{cases}$$

$$=\frac{2c}{l}\left[\int_0^{\frac{l}{2}} x\sin\frac{n\pi x}{l}\,dx+\int_{\frac{l}{2}}^{l} (l-x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$=\frac{2c}{l}\left\{\left[(x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_0^{\frac{l}{2}}+\left[(l-x)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(-1)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right)\right]_{\frac{l}{2}}^{l}\right\}$$

$$=\frac{2c}{l}\left\{\left[-\frac{l}{n\pi}x\cos\frac{n\pi x}{l}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^{\frac{L}{2}}+\left[-\frac{l}{n\pi}(l-x)\cos\frac{n\pi x}{l}-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_{\frac{L}{2}}^{L}\right\}$$

$$=\frac{2c}{l}\left\{\left[\left(-\frac{l}{n\pi}\frac{l}{2}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)-(0)\right]_0^{\frac{l}{2}}+\left[(0)-\left(-\frac{l}{n\pi}\left(\frac{l}{2}\right)\cos\frac{n\pi}{2}-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]_{\frac{l}{2}}^{l}\right\}$$

$$=\frac{2c}{l}\left[-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}+\frac{l^2}{2n\pi}\cos\frac{n\pi}{2}+\frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$=\frac{2c}{l}\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$b_n\frac{n\pi a}{l}=\frac{4cl}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad \because\ B_n=b_n\frac{n\pi a}{l}$$

$$\boxed{b_n=\frac{4cl^2}{n^3\pi^3 a}\sin\frac{n\pi}{2}}$$

Sub b_n in (4)

$$\therefore y(x,t)=\sum_{n=1}^{\infty}\frac{4cl^2}{n^3\pi^3 a}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

$$\therefore y(x,t)=\frac{4cl^2}{\pi^3 a}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\cos\frac{n\pi at}{l}$$

7. **If a string of length of *l* is initially at rest in its equilibrium position and each of its point is given the velocity $\frac{\partial y}{\partial t}(x,0)=V_0\sin^3\frac{\pi x}{l}$; *0<x<l*. Determine the displacement function *y(x,t)*.**

Solution:

One dimensional wave equation is $\frac{\partial^2 y}{\partial t^2} = a^2 \frac{\partial^2 y}{\partial x^2}$ where $a^2 = \frac{T}{m}$

The correct solution is

$$y(x,t) = (A\cos px + B\sin px)(C\cos pat + D\sin pat) \quad -----(1)$$

The Boundary and Initial conditions are

i) $y(0,t) = 0$

ii) $y(l,t) = 0$

iii) $y(x,0) = 0$

iv) $\frac{\partial y}{\partial t}(x,0) = V_0 \sin^3 \frac{\pi x}{l},\ 0 < x < l$

Applying condn (i) in (1)

$$(1) \Rightarrow y(0,t) = (A\cos 0 + \cancel{B\sin 0})(C\cos pat + D\sin pat)$$

$$0 = (A)(C\cos pat + D\sin pat)$$

Here $(C\cos pat + D\sin pat) \neq 0 \quad \therefore \boxed{A = 0}$

Sub A=0 in (1)

$$(1) \Rightarrow y(x,t) = (B\sin px)(C\cos pat + D\sin pat) \quad -----(2)$$

Applying condn (ii) in (2)

$$y(l,t) = (B\sin pl)(C\cos pat + D\sin pat)$$

$$0 = (B\sin pl)(C\cos pat + D\sin pat)$$

Here $B \neq 0,\ (C\cos pat + D\sin pat) \neq 0$

$$\therefore \sin pl = 0 \Rightarrow \sin pl = \sin n\pi \Rightarrow pl = n\pi \Rightarrow \boxed{p = \frac{n\pi}{l}}$$

Sub the value of p in (2)

$$(2) \Rightarrow \; y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(C\cos\frac{n\pi a}{l}t + D\sin\frac{n\pi a}{l}t\right) \quad -----(3)$$

Apply condn. (iii) in the above equation

$$(3) \Rightarrow \; y(x,0) = \left(B\sin\frac{n\pi}{l}x\right)\left(C\cos 0 + \cancel{D\sin 0}\right)$$

$$0 = \left(B\sin\frac{n\pi}{l}x\right)(C)$$

Here $B \neq 0$, $\sin\frac{n\pi}{l}x \neq 0$, $\therefore \boxed{C = 0}$

Sub the value of C in (3)

$$(3) \Rightarrow \; y(x,t) = \left(B\sin\frac{n\pi}{l}x\right)\left(D\sin\frac{n\pi a}{l}t\right)$$

$$y(x,t) = BD\sin\frac{n\pi}{l}x\sin\frac{n\pi a}{l}t$$

$$y(x,t) = b_1\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t \qquad \text{let } BD = b_1$$

The most general solution is

$$y(x,t) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{l}x\sin\frac{n\pi a}{l}t \quad -----(4)$$

Diff (4) partially w.r.to ‘t’

$$\frac{\partial y}{\partial t}(x,t) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{l}x\cos\frac{n\pi a}{l}t \times \left(\frac{n\pi a}{l}\right)$$

Applying condn (iv) in the above eqn.

$$\frac{\partial y}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n\sin\frac{n\pi}{l}x\cos 0 \times \left(\frac{n\pi a}{l}\right)$$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\times\left(\frac{n\pi a}{l}\right) \qquad \because \cos 0 = 1$$

$$V_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l}$$

$$V_0 \left\{\frac{1}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right]\right\} = \sum_{n=1}^{\infty} b_n \frac{n\pi a}{l} \sin\frac{n\pi x}{l} \qquad \because \sin^3\theta = \frac{1}{4}\left[3\sin\theta - \sin 3\theta\right]$$

$$\frac{3V_0}{4}\sin\frac{\pi x}{l} - \frac{V_0}{4}\sin\frac{3\pi x}{l} = b_1\frac{\pi a}{l}\sin\frac{\pi x}{l} + b_2\frac{2\pi a}{l}\sin\frac{2\pi x}{l} + b_3\frac{3\pi a}{l}\sin\frac{3\pi x}{l} + b_4\frac{4\pi a}{l}\sin\frac{4\pi x}{l} + \ldots$$

Equating co-efficients of likely terms on both sides

$$b_1\frac{\pi a}{l} = \frac{3V_0}{4};\ b_2\frac{2\pi a}{l} = 0;\ b_3\frac{3\pi a}{l} = \frac{-V_0}{4};\ b_4 = b_5 = b_6 \ldots = 0.$$

$$\boxed{b_1 = \frac{3V_0 l}{4\pi a};\ b_2 = 0;\ b_3 = \frac{-V_0 l}{12\pi a};\ b_4 = b_5 = b_6 \ldots = 0}.$$

Sub these values in (4)

$$(4) \Rightarrow y(x,t) = b_1\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} + b_2\sin\frac{2\pi x}{l}\cos\frac{2\pi at}{l} + b_3\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l} + \ldots.$$

$$\boxed{y(x,t) = \frac{3V_0 l}{4\pi a}\sin\frac{\pi x}{l}\cos\frac{\pi at}{l} - \frac{V_0 l}{12\pi a}\sin\frac{3\pi x}{l}\cos\frac{3\pi at}{l}}$$