2.1 FOURIER SERIES ANALYSIS

The Fourier representation of signals can be used to perform frequency domain analysis of signals in which we can study the various frequency components present in the signal, magnitude and phase of various frequency components.

Conditions for existence of Fourier series:

The Fourier series exist only if the following Dirichlet's conditions are satisfied.

- The signal $x(t)$ must be single valued function.
- The signal $x(t)$ must possess only a finite number of discontinuous in the period T.
- The signal must have a finite number of maxima and minima in the period T.
- $x(t)$ must be absolutely integrable.

$$\int_0^T |x(t)|dt < \infty$$

Types of Fourier series:

- Trigonometric Fourier series
- Exponential Fourier series
- Cosine Fourier series

TRIGONOMETRIC FOURIER SERIES

The trigonometric form of Fourier series of a periodic signal, $x(t)$ with period T is defined as

$$x(t) = a_o + \sum_{n=1}^{\infty} a_n \cos n\,\Omega_o t + \sum_{n=1}^{\infty} b_n \sin n\Omega_o t \text{ --------(1)}$$

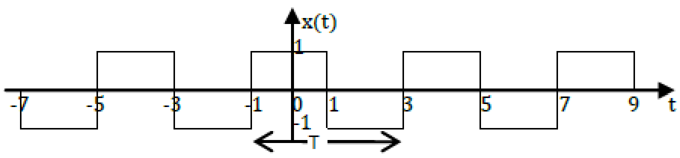
$where\ a_o, a_n, b_n$ → Fourier coefficients of trigonometric form of Fourier series

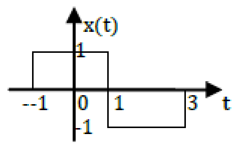
$$a_o = \frac{1}{T}\int_{t_o}^{t_o+T} x(t)dt$$

$$a_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)\cos n\,\Omega_o dt$$

$$b_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)\sin n\,\Omega_o dt$$

EXAMPLE 1: Find the trigonometric Fourier series for the periodic signal x(t) as shown in Figure





Solution:

$$T = 3-(-1) = 4 \; and \; \Omega_o = \frac{2\pi}{T} = \frac{\pi}{2}$$

To find a_o

$$a_o = \frac{1}{T}\int_{t_o}^{t_o+T} x(t)dt = \frac{1}{4}\left[\int_{-1}^{1} 1dt + \int_{1}^{3} -1dt\right]$$

$$= \frac{1}{4}[[t]_{-1}^{1} - 1[t]_{1}^{3}]$$

$$= \frac{1}{4}[2-2] = 0$$

To find a_n

$$\boldsymbol{a_n} = \frac{2}{T}\int_{t_0}^{t_0+T} x(t)\cos n\Omega_0 t\,dt = \frac{2}{4}\left[\int_{-1}^{1}\cos n\Omega_0 t\,dt + \int_{1}^{3}(-1)\cos n\Omega_0 t\,dt\right]$$

$$= \frac{1}{2}\left[\left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_{-1}^{1} - \left[\frac{\sin n\Omega_0 t}{n\Omega_0}\right]_{1}^{3}\right]$$

$$= \frac{1}{2}\left[\left[\frac{\sin n\,\frac{\pi t}{2}}{n\frac{\pi t}{2}}\right]_{-1}^{1} - \left[\frac{\sin n\,\frac{\pi t}{2}}{n\frac{\pi t}{2}}\right]_{1}^{3}\right]$$

$$= \frac{1}{2}\left(\frac{2}{n\pi}\right)\left[\sin n\,\frac{\pi}{2} - \left(\sin n\,\frac{\pi}{2}(-1)\right) - \left(\sin n\,\frac{\pi}{2}(3) - \sin n\,\frac{\pi}{2}\right)\right]$$

$$= \left[\frac{1}{n\pi}\right]\left[\sin n\,\frac{\pi}{2} + \sin n\,\frac{\pi}{2} - \sin 3n\,\frac{\pi}{2} + \sin n\,\frac{\pi}{2}\right]$$

$$= \frac{1}{n\pi}\left[3sin\frac{n\pi}{2} - \left(-\sin n\,\frac{\pi}{2}\right)\right] = \frac{4}{n\pi}\left[\sin n\,\frac{\pi}{2}\right]$$

To find b_n

$$b_n = \frac{2}{T}\int_{t_o}^{t_o+T} x(t)\sin n\,\Omega_o dt$$

$$= \frac{2}{4}\left[\int_{-1}^{1} \sin n\,\Omega_o t dt + \int_{1}^{3} -\sin n\,\Omega_o t dt\right]$$

$$= \frac{1}{2}\left[\left[\frac{-\cos n\,\Omega_o t}{n\,\Omega_o}\right]_{-1}^{1} - \left[\frac{-\cos n\,\Omega_o t}{n\,\Omega_o}\right]_{1}^{3}\right] = \frac{1}{2}\left[\left[\frac{-\cos n\,\frac{\pi}{2}t}{n\,\frac{\pi}{2}}\right]_{-1}^{1} + \left[\frac{\cos n\,\frac{\pi}{2}t}{n\,\frac{\pi}{2}}\right]_{1}^{3}\right]$$

$$= \frac{1}{2}\left[\frac{-2}{n\pi}\left(\cos n\,\frac{\pi}{2} - \cos n\,\frac{\pi}{2}(-1)\right) + \frac{2}{n\pi}\left(\cos n\,\frac{\pi}{2}(3) - \cos n\,\frac{\pi}{2}\right)\right]$$

$$= \frac{1}{2}\left[0 + \frac{2}{n\pi}\left(\cos\left(2n\pi - \frac{n\pi}{2}\right) - \cos n\,\frac{\pi}{2}\right)\right]$$

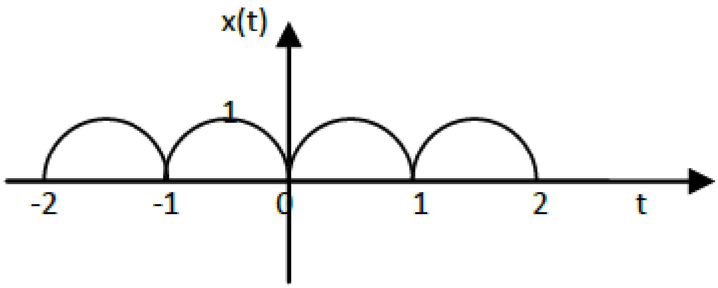
$$= \left[\frac{1}{n\pi}\left(\cos n\,\frac{\pi}{2} - \cos n\,\frac{\pi}{2}\right)\right] = 0$$

Trigonometric Fourier Series

$$\boldsymbol{x(t) = a_o + \sum_{n=1}^{\infty} a_n \cos n\,\Omega_o t + \sum_{n=1}^{\infty} b_n \sin n\Omega_o t}$$

$$\boldsymbol{= \sum_{n=1}^{\infty} \frac{4}{n\pi} sin\left(\frac{n\pi}{2}\right)\cos n\,\Omega_o t = \sum_{n=1}^{\infty} \frac{4}{n\pi} sin\left(\frac{n\pi}{2}\right)\cos n\,\frac{\pi}{2}}$$

EXAMPLE:2 Obtain Fourier series of the following full wave rectified sine wave shown in figure



Solution:

$$\boldsymbol{x(t) = x(-t); Given\ signal\ is\ even\ signal\ , so\ b_n = 0}$$

$$T = 1\ and\ \Omega_o = \frac{2\pi}{1} = 2\pi$$

To find a_o

$$\boldsymbol{a_0} = \frac{2}{T}\int_0^{\frac{T}{2}} x(t)\,dt = \frac{2}{1}\int_0^{\frac{1}{2}} x(t)\,dt = \left[2\int_0^{\frac{1}{2}} \sin\pi t\ \,dt\right]$$

$$= 2\left[-\frac{\cos\pi t}{\pi}\right]_0^{\frac{1}{2}} = -\frac{2}{\pi}\left[\cos\frac{\pi}{2} - \cos 0\right] = \frac{2}{\pi}$$

To find a_n

$$a_n = \frac{4}{T}\int_0^{\frac{T}{2}} x(t)\cos n\Omega_o t dt = \frac{4}{1}\int_0^{\frac{1}{2}} \sin\pi t\cos n2\pi t\, dt$$

$$= 2\int_0^{\frac{1}{2}}\left[sin\big((1+2n)\pi t\big) + sin\big((1-2n)\pi t\big)\right]dt$$

$$= 2\left[-\frac{\cos\big((1+2n)\pi t\big)}{(1+2n)\pi} - \frac{\cos\big((1-2n)\pi t\big)}{(1-2n)\pi}\right]_0^{\frac{1}{2}}$$

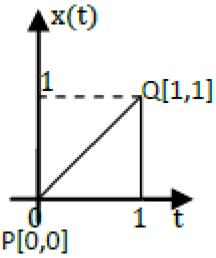
$$= \frac{2}{\pi}\left[-\frac{\cos\left((1+2n)\frac{\pi}{2}\right)}{1+2n} - \frac{\cos\left((1-2n)\frac{\pi}{2}\right)}{1-2n} + \frac{1}{1+2n} + \frac{1}{1-2n}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{1+2n} + \frac{1}{1-2n}\right] = \frac{2}{\pi}\left[\frac{1-2n+1+2n}{1-4n^2}\right] = \frac{4}{\pi(1-4n^2)}$$

Trigonometric Fourier Series

$$\boldsymbol{x(t) = a_o + \sum_{n=1}^{\infty} a_n \cos n\,\Omega_o t + \sum_{n=1}^{\infty} b_n \sin n\Omega_o t}$$

$$\boldsymbol{x(t)} = \frac{2}{\pi} + \sum_{n=1}^{\infty}\frac{4}{\pi(1-4n^2)}\cos n2\pi t$$



NTIAL FOURIER SERIES

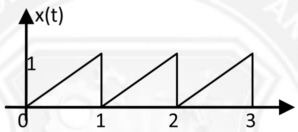
The exponential form of Fourier series of a periodic signal $x(t)$ with period T is defined as,

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\,\Omega_0 t}$$

The Fourier coefficient C_n can be evaluated using the following formulae

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jn\,\Omega_0 t} dt$$

EXAMPLE 3: Find exponential series for the signal shown in figure

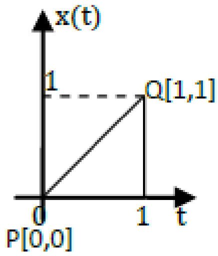


Solution:

$$T = 1, \Omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi$$

Consider the equation of a straight line

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$



Consider points P,Q as shown in figure

Coordinates of point P = [0,0]

Coordinates of point Q = [1,1]

On substituting the coordinates of points P and Q in equation

$$\frac{x(t)-0}{1-0}=\frac{t-0}{1-0}\Rightarrow x(t)=t$$

$$x=t,y=x(t)$$

To find C_o

$$c_0=\frac{1}{T}\int_0^T x(t)dt=\frac{1}{1}\int_0^1 (t)dt$$

$$=\left[\frac{t^2}{2}\right]_0^1=\frac{1}{2}$$

To find C_n

$$c_n=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jn\Omega_0 t}dt$$

$$=\frac{1}{1}\int_0^1 te^{-jn2\pi t}dt=\left[t\frac{e^{-jn2\pi t}}{-jn2\pi}\right]_0^1-\int_0^1\frac{e^{-jn2\pi t}}{-jn2\pi}dt$$

$$=\frac{e^{-j2\pi n}}{-j2\pi n}+0+\left[\frac{e^{-j2\pi nt}}{-j^2(2\pi n)^2}\right]_0^1$$

$$=j\frac{e^{-jn2\pi}}{n2\pi}+\frac{e^{-jn2\pi}}{n^2 4\pi^2}-\frac{1}{n^2 4\pi^2}$$

$$=\frac{j}{n2\pi}+\frac{1}{n^2 4\pi^2}-\frac{1}{n^2 4\pi^2}=\frac{j}{n2\pi}$$

$$c_n=\frac{j}{n2\pi}$$

$$C_1=\frac{j}{2\pi}, \qquad C_2=\frac{j}{4\pi},$$

$$c_{-1}=\frac{j}{-2\pi}, \qquad c_{-2}=\frac{j}{-4\pi}, \qquad c_{-3}=\frac{j}{-6\pi}$$

Exponential Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\Omega_0 t}$$

$$\therefore\ x(t) = + \cdots - \frac{j}{6\pi} e^{-j6\pi t} - \frac{j}{4\pi} e^{-j4\pi t} - \frac{j}{2\pi} e^{-j2\pi t} + \frac{1}{2} + \frac{j}{2\pi} e^{j2\pi t} + \frac{j}{4\pi} e^{j4\pi t} + \frac{j}{6\pi} e^{j6\pi t} + \cdots$$

$$= \frac{1}{2} + \frac{j}{2\pi}\left[e^{j2\pi t} - e^{-j2\pi t}\right] + \frac{j}{4\pi}\left[e^{j4\pi t} - e^{-j4\pi t}\right] + \frac{j}{6\pi}\left[e^{j6\pi t} - e^{-j6\pi t}\right] + \cdots$$

$$= \frac{1}{2} + \frac{1}{\pi}\left[\frac{e^{j2\pi t} - e^{-j2\pi t}}{(-1)2j}\right] + \frac{1}{2\pi}\left[\frac{e^{j4\pi t} - e^{-j4\pi t}}{(-1)2j}\right] + \frac{1}{3\pi}\left[\frac{e^{j6\pi t} - e^{-j6\pi t}}{(-1)2j}\right]$$

$$= \frac{1}{2} + \left(\frac{-1}{\pi}\right) \sin 2\pi t - \frac{1}{2\pi} \sin 4\pi t - \frac{1}{3\pi} \sin 6\pi t$$

$$= \frac{1}{2} - \frac{1}{\pi}\left[\sin 2\pi t + \frac{1}{2} \sin 4\pi t + \frac{1}{3} \sin 6\pi t + \cdots\right]$$

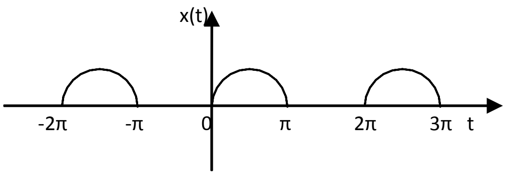
COSINE FOURIER SERIES

Cosine representation of $x(t)$ is

$$x(t) = A_o + \sum_{n=1}^{\infty} A_n \cos(n\Omega_o t + \theta_n)$$

Where A_0 is dc component, A_n is harmonic amplitude or spectral amplitude and θ_n is phase coefficient or phase angle *or spectral angle*

EXAMPLE 4: Determine the cosine Fourier series of the signal shown in Figure



Solution:

The signal shown in is periodic with period

$$T = 2\pi \ \ and\ \Omega_o = \frac{2\pi}{2\pi} = 1$$

The given signal is sinusoidal signal,

$$\therefore x(t) = A \sin \Omega t$$

Here, $\Omega = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1, \ \ A = 1$

$$\therefore \boldsymbol{x(t) = sin\, t}$$

To find a_o

$$a_o = \frac{1}{T}\int_0^T x(t)dt$$

$$= \frac{1}{2\pi}\int_0^{\pi} sin\, t\, dt = \frac{1}{2\pi}[-\cos t\,]_0^{\pi} = \frac{1}{2\pi}[-\cos\pi + \cos 0] = \frac{1}{2\pi}[2] = \frac{1}{\pi}$$

To find a_n

$$a_n = \frac{2}{T}\int_0^T x(t)\cos n\Omega_0 t\, dt = \frac{2}{2\pi}\int_0^{\pi} sin\, t\ \cos nt\, dt = \frac{1}{2\pi}\int_0^{\pi}[sin(1+n)t + sin(1-n)t]\, dt$$

$$= \frac{1}{2\pi}\left[-\frac{cos(1+n)t}{(1+n)} - \frac{cos(1-n)t}{(1-n)}\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{cos(1+n)\pi}{(1+n)} - \frac{cos(1-n)\pi}{(1-n)} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$for\ n = odd:\ a_n = \frac{1}{2\pi}\left[-\frac{1}{1+n} - \frac{1}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right] = 0$$

$$for\ n = even:\ a_n = \frac{1}{2\pi}\left[\frac{1}{1+n} + \frac{1}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-n+1+n}{1-n^2}\right] = \frac{2}{\pi(1-n^2)}$$

$$a_n = \begin{cases} 0 & for\ n = odd \\ \dfrac{2}{\pi(1-n^2)} & for\ n = even \end{cases}$$

To find b_n

$$\boldsymbol{b_n} = \frac{2}{T}\int_0^T x(t)\sin n\Omega_0 t\ dt = \frac{2}{2\pi}\int_0^\pi \sin t\ \sin nt\ dt$$

$$= \frac{1}{2\pi}\int_0^\pi (\cos(1-n)t - \cos(1+n)t)\,dt$$

$$= \frac{1}{2\pi}\left[\frac{\sin(1-n)t}{(1-n)} - \frac{\sin(1+n)t}{(1+n)}\right]_0^\pi$$

$$= \frac{1}{2\pi}\left[\frac{\sin(1-n)\pi}{(1-n)} - \frac{\sin(1+n)\pi}{(1+n)} - 0\right] = 0$$

calculate the Fourier coefficients of Cosine Fourier series from Trigonometric Fourier series:

$$A_0 = a_0 = \frac{1}{\pi}$$

$$A_n = \sqrt{a_n^2 + b_n^2} = \frac{2}{\pi(1-n^2)},for\ n\ even$$

$$\theta_n = -tan^{-1}\frac{b_n}{a_n} = 0$$

Cosine Fourier Series

$$\mathrm{x(t)} = \mathrm{A_0} + \sum_{\mathrm{n}=1}^{\infty} \mathrm{A_n}\cos(\mathrm{n}\Omega_0\mathrm{t} + \theta_\mathrm{n})$$

$$x(t) = \frac{1}{\pi} + \sum_{\substack{n=1 \\ (n=even)}}^{\infty} \frac{2}{\pi(1-n^2)} \cos nt$$

$$= \frac{1}{\pi} + \frac{2}{\pi(1-4)} cos\ 2t + \frac{2}{\pi(1-16)} cos\ 4t + \cdots$$

$$= \frac{1}{\pi} - \frac{2}{3\pi} cos\ 2t - \frac{2}{15\pi} cos\ 4t + \cdots = \frac{1}{\pi} - \frac{2}{\pi}[\frac{1}{3} cos\ 2t + \frac{1}{15} cos\ 4t + \cdots\ ]$$