UNIT – III

APPLICATIONS OF PARTIAL DIFFERENTIALEQUATIONS

Solution of Laplace's equation(Two dimentional heat equation)

The Laplace equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Let u = X(x) . Y(y) be the solution of (1), where „X" is a function of „x" alone and „Y" isa function of „y" alone.

Then $\frac{\partial^2 u}{\partial x^2} = X'' Y$ and $\frac{\partial^2 u}{\partial y^2} = . X Y''$

Substituting in (1), we have

$$X'' Y + X Y'' = 0$$

i.e, $\frac{X''}{X} = -\frac{Y''}{Y}$ -------------------------------- (2).

Now the left side of (2) is a function of „x" alone and the right side is a function of „t"alone. Since „x" and „t" are independent variables, (2) can be true only if each side is equal to a constant.

Therefore, $\frac{X''}{X} = -\frac{Y''}{Y} = k$ (say).

Hence, we get $X'' - kX = 0$ and $Y'' + kY = 0$ --------------------------- (3).

Solving equations (3), we get

(i) when „k‟ is positive and k = λ^2, say
(ii)

$$X = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$$

$$Y = c_3 \cos\lambda y + c_4 \sin\lambda y$$

(iii) when „k‟ is negative and k = $-\lambda^2$, say
(iv)

$$X = c_5 \cos\lambda x + c_6 \sin\lambda x Y$$
$$= c_7 e^{\lambda y} + c_8 e^{-\lambda y}$$

(v) when „k‟ is zero.

$X = c_9\ x + c_{10}Y$
$= c_{11}\,x + c_{12}$

Thus the various possible solutions of (1) are

$$u = (c_1 e^{\lambda x} + c_2\ e^{-\lambda x})\,(c_3 \cos\lambda y + c_4 \sin\lambda y) \text{-----------------------} (4)$$
$$u = (c_5 \cos\lambda x + c_6 \sin\lambda x)\,(c_7 e^{\lambda y} + c_8\ e^{-\lambda y}) \text{-----------------------} (5)$$
$$u = (c_9\ x + c_{10})\,(c_{11}\,x + c_{12}) \text{---} (6)$$

Of these three solutions, we have to choose that solution which suits the physical nature of the problem and the given boundary conditions.

Example 12

An infinitely long uniform plate is bounded by two parallel edges x = 0 & x = ℓ and an end at right angles to them. The breadth of this edge y = 0 is ℓ and this edge is maintained at a temperature f (x). All the other 3 edges are at temperature zero. Find thesteady state temperature at any interior point of the plate.

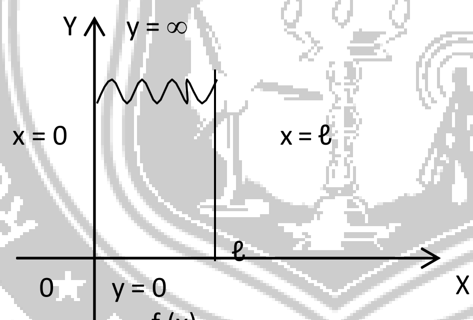
Solution

Let u (x,y) be the temperature at any point x,y of the plate.

Also u (x,y) satisfies the equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ -------------------------------------- (1)

Let the solution of equation (1) be

$$u(x,y) = (A \cos \lambda x + B \sin\lambda x)\ (Ce^{\lambda y} + De^{-\lambda y}) \text{--} (2)$$



0 < x < ℓ0
< y < ∞

The boundary conditions are

(i) u (0, y) = 0, for $0 < y < \infty$
(ii) u (ℓ, y) = 0, for $0 < y < \infty$
(iii) u (x, ∞) = 0, for $0 < x < ℓ$
(iv) u (x, 0) = f(x), for $0 < x < ℓ$

Using condition (i), we get 0 =

$$A\,(Ce^{\lambda y} + De^{-\lambda y})$$

i.e, A = 0

$\therefore$ Equation (2) becomes,

$$u(x,y) = B \sin\lambda x\,(Ce^{\lambda y} + De^{-\lambda y}) \quad \text{------ (3)}$$

Using cndition (ii), we get

$$\lambda = \frac{n\pi}{\ell}$$

Therefore, $$u(x,y) = B \sin\frac{n\pi x}{\ell}\left\{Ce^{(n\pi y/\ell)} + De^{(-n\pi y/\ell)}\right\} \quad \text{------ (4)}$$

Using condition (iii), we get C = 0.

$$\therefore u(x,y) = B \sin\frac{n\pi x}{\ell}\, De^{(-n\pi y/\ell)}$$

i.e, $u(x,y) = B_1 \sin\frac{n\pi x}{\ell}\, e^{(-n\pi y/\ell)}$, where $B_1 = BD$.

The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell}\, e^{(-n\pi y/\ell)} \quad \text{------ (5)}$$

Using condition (iv), we get

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{\ell} \quad \text{------ (6)}$$

The RHS of equation (6) is a half – range Fourier sine series of the LHS function.

$$\therefore B_n = \frac{2}{\ell}\int_0^{\ell} f(x).\sin\frac{n\pi x}{\ell}\,dx \quad \text{------ (7)}$$

Using (7) in (5), we get the required solution.

Example 13

A rectangular plate with an insulated surface is 8 cm. wide and so long comparedto its width that it may be considered as an infinite plate. If the temperature along short edge y = 0 is u(x,0) = 100 sin (πx/8), $0 \le x \le 8$, while two long edges x = 0 & x = 8 as well as the other short edges are kept at 0°C. Find the steady state temperature at any point of the plate.

Solution

The two dimensional heat equation is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{-------------- (1)}$$

The solution of equation (1) be

$$u(x,y) = (A\cos\lambda x + B\sin\lambda x)\,(Ce^{\lambda y} + De^{-\lambda y}) \quad \text{------ (2)}$$

The boundary conditions are

(i) u (0, y) = 0, for $0 < y < \infty$
(ii) u (8, y) = 0, for $0 < y < \infty$
(iii) u (x, ∞) = 0, for $0 < x < 8$
(iv) u (x, 0) = 100 Sin (πx/8,) for $0 < x < 8$

Using conditions (i), & (ii), we get

$$A = 0,\ \lambda = \frac{n\pi}{8}$$

$$\therefore u(x,y) = B\sin\frac{n\pi x}{8}\left(Ce^{(n\pi y/8)} + De^{(-n\pi y/8)}\right)$$

$$= \left(B_1e^{(n\pi y/8)} + D_1e^{(-n\pi y/8)}\right)\sin\frac{n\pi x}{8}, \text{ where } B_1 = BC,\ D_1 = BD$$

The most general soln is

$$u(x,y) = \sum_{n=1}^{\infty}\left(B_ne^{(n\pi y/8)} + D_ne^{(-n\pi y/8)}\right)\sin\frac{n\pi x}{8} \quad \text{------ (3)}$$

Using condition (iii), we get $B_n = 0$.

Hence, $u(x,y) = \sum_{n=1}^{\infty} D_n e^{(-n\pi y/8)} \sin \frac{n\pi x}{8}$ -- (4)

Using condition (iv), we get

$$100 \sin \frac{\pi x}{8} = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{8}$$

$$\text{i.e, } 100 \sin \frac{\pi x}{8} = D_1 \sin \frac{\pi x}{8} + D_2 \sin \frac{2\pi x}{8} + D_3 \sin \frac{3\pi x}{8} + \ldots$$

Comparing like coefficients on both sides, we get $D_1 =$

100, $D_2 = D_3 = \ldots\ldots\ldots\ldots = 0$

Substituting in (4), we get

$$u(x,y) = 100\, e^{(-\pi y/8)} \sin(\pi x/8)$$

Example 14

A rectangular plate with an insulated surface 10 c.m wide & so long compared toits width that it may considered as an infinite plate. If the temperature at the short edge y = 0 is given by

$$u(x,0) = \begin{cases} 20x, & 0 \le x \le 5 \\ 20(10-x), & 5 \le x \le 10 \end{cases}$$

and all the other 3 edges are kept at temperature 0°C. Find the steady state temperature atany point of the plate.

Solution

The temperature function u (x,y) is given by the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{.........................(1)}$$

The solution is

$$u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y}) \quad \text{---------------- (2)}$$

The boundary conditions are

(i) $u(0, y) = 0$, for $0 \le y \le \infty$

(ii) $u(10, y) = 0$, for $0 \le y \le \infty$

(iii) $u(x, \infty) = 0$, for $0 \le x \le 10$

(iv) $u(x, 0) = \begin{cases} 20x, & \text{if } 0 \le x \le 5 \\ 20(10-x), & \text{if } 5 \le x \le 10 \end{cases}$

Using conditions (i), (ii), we get

$$A = 0 \;\&\; \lambda = \frac{n\pi}{10}$$

$\therefore$ Equation (2) becomes

$$u(x,y) = B \sin \frac{n\pi x}{10} \left(Ce^{(n\pi y/10)} + De^{(-n\pi y/10)} \right)$$

$$= \left(B_1 e^{(n\pi y/10)} + D_1 e^{(-n\pi y/10)} \right) \sin \frac{n\pi x}{10} \quad \text{where } B_1 = BC,\; D_1 = BD$$

$\therefore$ The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty} \left(B_n e^{(n\pi y/10)} + D_n e^{(-n\pi y/10)} \right) \sin \frac{n\pi x}{10} \quad \text{------------ (3)}$$

Using condition (iii), we get $B_n = 0$.

$\therefore$ Equation (3) becomes

$$u(x,y) = \sum_{n=1}^{\infty} D_n e^{(-n\pi y/10)} \sin \frac{n\pi x}{10} \quad \text{------------ (4)}$$

Using condition (iv), we get

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin \frac{n\pi x}{10} \quad \text{------- (5)}$$

The RHS of equation (5) is a half range Fourier sine series of the LHS function

$$\therefore D_n = \frac{2}{10}\int_0^{10} f(x) \sin\frac{n\pi x}{10}\, dx$$

$$= \frac{2}{10}\left\{ \left[(20x)\left(\frac{\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}} \right) - (20)\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}} \right) \right]_0^5 + \left[20(10-x)\left(\frac{-\cos\frac{n\pi x}{10}}{\frac{n\pi}{10}} \right) - (-20)\left(\frac{-\sin\frac{n\pi x}{10}}{\frac{n^2\pi^2}{100}} \right) \right]_5^{10} \right\}$$

i.e, $$D_n = \frac{800 \sin\frac{n\pi}{2}}{n^2\pi^2}$$

Substituting in (4) we get,

$$u(x,y) = \sum_{n=1}^{\infty} \frac{800 \sin\frac{n\pi}{2}}{n^2\pi^2}\, e^{(-n\pi y/10)} \sin\frac{n\pi x}{10}$$

Example 15

A rectangular plate is bounded by the lines x = 0, x = a, y = 0 & y = b.

The edge temperatures are u (0,y) = 0, u (x,b) = 0, u (a,y) = 0 &

$u(x,0) = 5 \sin(5\pi x / a) + 3 \sin(3\pi x / a)$. Find the steady state temperature distribution atany point of the plate.

The temperature function u (x,y) satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{---------- (1)}$$

Let the solution of equation (1) be

$$u(x,y) = (A\cos\lambda x + B\sin\lambda x)(Ce^{\lambda y} + De^{-\lambda y}) \text{------------------------------------ (2)}$$

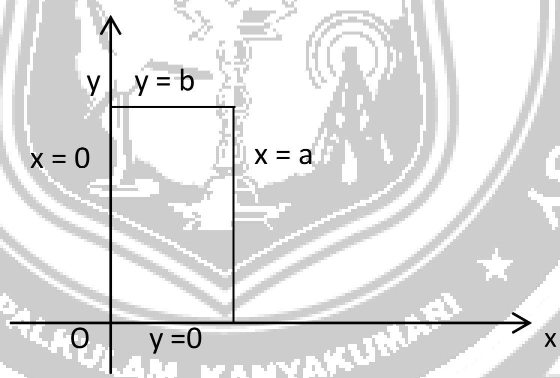
The boundary conditions are

(i) $u(0,y) = 0$, for $0 < y < b$

(ii) $u(a,y) = 0$, for $0 < y < b$

(iii) $u(x,b) = 0$, for $0 < x < a$

(iv) $u(x,0) = 5 \sin(5\pi x / a) + 3 \sin(3\pi x / a)$, for $0 < x < a$.



Using conditions (i), (ii), we get

$$A = 0, \quad \lambda = \frac{n\pi}{a}$$

$$\therefore u(x,y) = B \sin\frac{n\pi x}{a}\left(Ce^{(n\pi y/a)} + De^{(-n\pi y/a)}\right)$$

$$= \sin\frac{n\pi x}{a}\left(B_1 e^{(n\pi y/a)} + D_1 e^{(-n\pi y/a)}\right)$$

The most general solution is

$$u(x,y) = \sum_{n=1}^{\infty}\left(B_n e^{(n\pi y/a)} + D_n e^{(-n\pi y/a)}\right)\sin\left(\frac{n\pi x}{a}\right) \text{-----(3)}$$

Using condition (iii) we get

$$0 = \sum_{n=1}^{\infty} \left(B_n e^{(n\pi b / a)} + D_n e^{(-n\pi b / a)} \right) \sin \frac{n\pi x}{a}$$

$$\Longrightarrow B_n e^{(n\pi b / a)} + D_n e^{(-n\pi b / a)} = 0$$

$$\therefore \quad D_n = B_n \frac{e^{(n\pi b / a)}}{-e^{(-n\pi b / a)}} = - B_n e^{(2n\pi b / a)}$$

Substituting in (3), we get

$$u(x,y) = \sum_{n=1}^{\infty} \left(B_n e^{(n\pi y / a)} - B_n e^{(2n\pi b / a)} e^{(-n\pi y / a)} \right) \sin \frac{n\pi x}{a}$$

$$= \sum_{n=1}^{\infty} \frac{B_n}{e^{(-n\pi b)/a}} \left(e^{(n\pi y / a)} e^{(-n\pi b / a)} - e^{(2n\pi b / a)} e^{(-n\pi y / a)} e^{(-n\pi b / a)} \right) \sin \frac{n\pi x}{a}$$

$$= \sum \frac{2 B_n}{e^{(-n\pi b / a)}} \sin \frac{n\pi x}{a}$$

$$= \sum \frac{2B_n}{e^{(-n\pi b / a)}} \sin h \frac{n\pi (y-b)}{a} \sin \frac{n\pi x}{a}$$

$$\text{i.e, } u(x,y) = \sum_{n=1}^{\infty} C_n \sin h \frac{n\pi (y-b)}{a} \sin \frac{n\pi x}{a} \quad \text{------ (4)}$$

Using condition (iv), we get

$$\sin \frac{5\pi x}{a} + 3 \sin \frac{3\pi x}{a} = \sum_{n=1}^{\infty} C_n \sin h \frac{n\pi (-b)}{a} \sin \frac{n\pi x\, 5}{a}$$

$$\text{ie, } 5 \sin \frac{5\pi x}{a} + 3 \sin \frac{3\pi x}{a} = \sum_{n=1}^{\infty} - C_n \sin h \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$$

ie, $5 \sin \frac{5\pi x}{a} + 3 \sin \frac{3\pi x}{a} = - C_1 \sinh \frac{\pi b}{a} \sin \frac{\pi x}{a} - C_2 \sin h \frac{2\pi b}{a} \sin \frac{2\pi x}{a} - \ldots$

Comparing the like coefficients on both sides, we get

$$- C_3 \sinh \frac{3\pi b}{a} = 3 \quad \&$$

$$- C_5 \sinh \frac{5\pi b}{a} = 5, \qquad C_1 = C_2 = C_4 = C_6 = \ldots = 0$$

$$\Longrightarrow C_3 = \frac{-3}{\sinh(3\pi b/a)} \;\&\; C_5 = \frac{-5}{\sinh(5\pi b/a)}$$

Substituting in (4), we get

$$u(x,y) = - \frac{3}{\sinh(3\pi b/a)} \sin h \frac{3\pi (y-b)}{a} \sin \frac{3\pi x}{a} - \frac{5}{\sinh(5\pi b/a)} \sin h \frac{5\pi (y-b)}{a} \sin \frac{5\pi x}{a}$$

i.e,
$$u(x,y) = \frac{3}{\sinh(3\pi b/a)} \sin h \frac{3\pi (b-y)}{a} \sin \frac{3\pi x}{a} + \frac{5}{\sinh(5\pi b/a)} \sin h \frac{5\pi (b-y)}{a} \sin \frac{5\pi x}{a}$$

Exercises

(1) Solve the Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, subject to the conditions

i. $u(0,y) = 0$ for $0 < y < b$

ii. $u(a,y) = 0$ for $0 < y < b$
iii. $u(x,b) = 0$ for $0 < x < a$
iv. $u(x,0) = \sin^3(\pi x/a)$, $0 < x < a$.

(2) Find the steady temperature distribution at points in a rectangular plate with insulatedfaces and the edges of the plate being the lines x = 0, x = a, y = 0 and y = b. When three of the edges are kept at temperature zero and the fourth at a fixed temperature α^o C.

(3) Solve the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, which satisfies the conditions $u(0,y) = u(l,y) = u(x,0) = 0$ and $u(x,a) = \sin(n\pi x/l)$.

(4) Solve the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, which satisfies the conditions $u(0,y) = u(a,y) = u(x,b) = 0$ and $u(x,0) = x(a - x)$.

(5) Solve the Laplace equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, subject to the conditions

i. $u(0,y) = 0$, $0 \le y \le l$ ii. $u(l,y) = 0$, $0 \le y \le l$
iii. $u(x,0) = 0$, $0 \le x \le l$ iv. $u(x,l) = f(x)$, $0 \le x \le l$

(6) A square plate is bounded by the lines x = 0, y = 0, x = 20 and y = 20. Its faces are insulated.
The temperature along the upper horizontal edge is given by $u(x,0) = x(20 - x)$, when $0 < x < 20$,
while other three edges are kept at 0^o C. Find the steady state temperature in the plate.

(7) An infinite long plate is bounded plate by two parallel edges and an end at right angles to them.The breadth is π. This end is maintained at a constant temperature „u_0‟ at all points and the other edges are at zero temperature. Find the steady state temperature atany point (x,y) of the plate.

(8) An infinitely long uniform plate is bounded by two parallel edges x = 0 and x = l, and an end at right angles to them. The breadth of this edge y = 0 is „l‟ and is maintained at a temperature f(x). All the other three edges are at temperature zero. Find the steady state temperature at any interior point of the plate.

(9) A rectangular plate with insulated surface is 8 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge y = 0 is given by $u(x,0) = 100 \sin(\pi x/8)$, $0 < x < 8$, while the two long edges x = 0 and x = 8 as well as the other short edge are kept at 0^o C, show that the steady state temperature at any point of the plane is given by $u(x,y) = 100\, e^{-\pi y/8} \sin \pi x/8$.

(10) A rectangular plate with insulated surface is 10 cm. wide and so long compared to its width that it may be considered infinite length. If the temperature along short edge y =0 is given

$u(x,0) = 8 \sin(\pi x/ 10)$ when $0 < x < 10$, while the two long edges $x = 0$ and $x = 10$ as well as the other short edge are kept at 0^o C, find the steady state temperature distributionu(x,y).