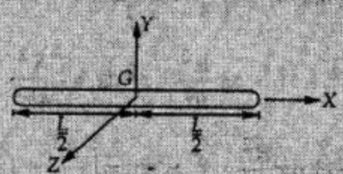
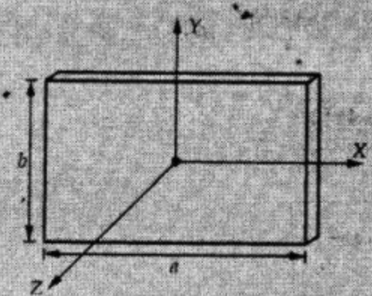
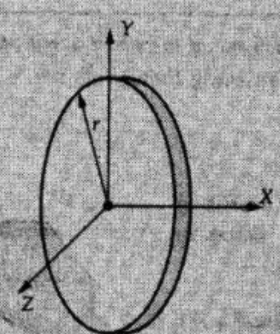
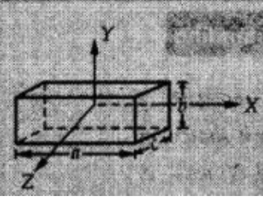
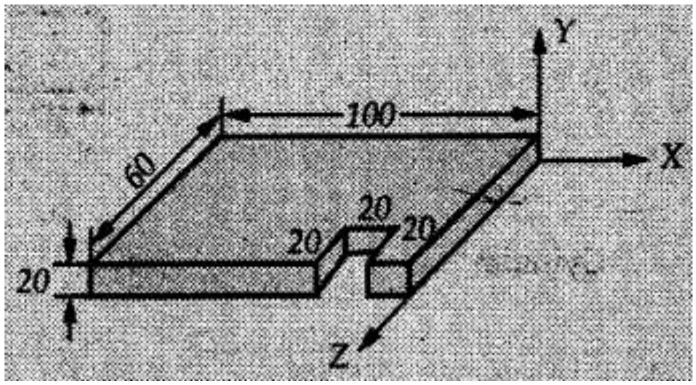
3.5 Mass Moment of Inertia of Composite Bodies

• The mass moment of inertia of a composite body about any axis can be obtained by first finding the moments of inertia of its component parts about that axis and then adding them.

Sr. No.	Standard shape		Mass moment of inertia
	Description	Figure	
1.	Slender rod		$I_Y = I_Z = \frac{mL^2}{12}$
2.	Thin rectangular plate		$I_X = \frac{mb^2}{12}$ $I_Y = \frac{ma^2}{12}$ $I_Z = \frac{m}{12}(a^2+b^2)$
3.	Thin disc		$I_X = I_Y = \frac{mr^2}{4}$ $I_Z = \frac{mr^2}{2}$
4.	Rectangular prism		$I_X = \frac{m}{12}(b^2+c^2)$ $I_Y = \frac{m}{12}(a^2+c^2)$ $I_Z = \frac{m}{12}(a^2+b^2)$

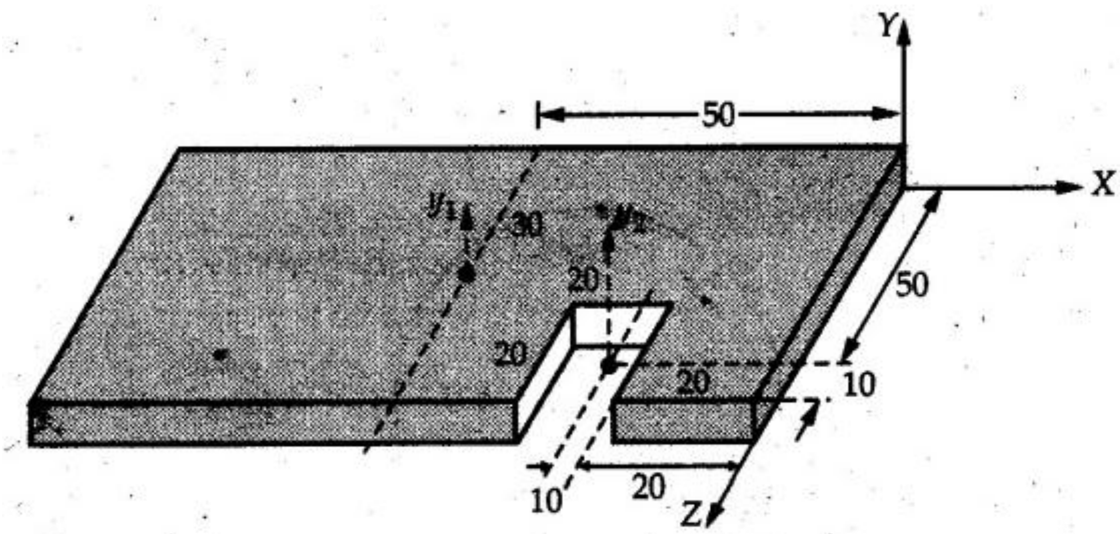
Solved Example

Find the mass moment of inertia of the rectangular block shown in Fig. about the vertical Y axis. A cuboid of 20 × 20 × 20 mm has been removed from the block as shown in the Fig. The mass density of the material of the block is 7850 kg/m^3.



Solution:

The centroidal Y-axes of the rectangular block and cuboid are shown in Fig. (a).



For rectangular block,

m_1 = 7850 × (0.1× 0.06 × 0.02) = 0.942 kg

Distance between Y-axis and y_1 axis,

$$d_1 = \sqrt{0.05^2 + 0.03^2}$$

$$\therefore \quad d_1^2 = 0.05^2 + 0.03^2$$

Moment of inertia about Y-axis is,

$$I_1 = \frac{0.942 \times (0.1^2 + 0.06^2)}{12} + 0.942 \times (0.05^2 + 0.03^2)$$

$$\therefore \quad I_1 = 4.2704 \times 10^{-3} \text{ kg-m}^2$$

For cuboid,

$m_2 = 7850 \times (0.02 \times 0.02 \times 0.02) = 0.0628$ kg

Distance between Y-axis and y_2 axis is

$$d_2 = \sqrt{0.05^2 + 0.03^2}$$

$$\therefore \quad d_2^2 = 0.05^2 + 0.03^2$$

$$\therefore \quad I_2 = \frac{0.0628(0.02^2 + 0.02^2)}{12} + 0.0628 \times (0.05^2 + 0.03^2)$$

$$\therefore \quad I_2 = 2.1771 \times 10^{-4} \text{ kg-m}^2$$

$$I_Y = I_1 - I_2 = 4.2704 \times 10^{-3} - 2.1771 \times 10^{-4}$$

$$\therefore \quad \boxed{I_Y = 4.0527 \times 10^{-3} \text{ kg-m}^2}$$

Parallel Axis Theorem

Consider an element of mass dm having coordinates x,y,z with respect to origin as shown in Fig (a). Then,

$$I_X = \int (y^2 + z^2)dm$$

$$I_Y = \int (x^2 + z^2)dm$$

and

$$I_Z = \int (x^2 + y^2)dm$$

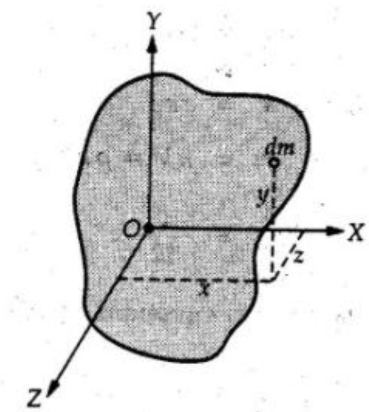


Figure (a)

Now consider two sets of parallel coordinate axes one passing through a point O and the other passing through the centre of gravity G of the body as shown in Fig. b

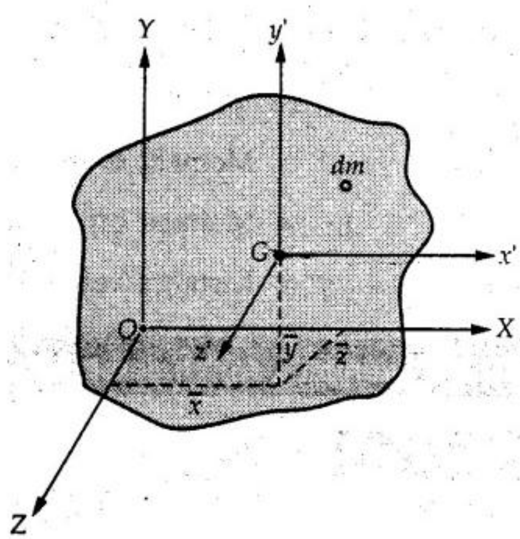


Figure (b)

Let $\bar{x}$, $\bar{y}, \bar{z}$ be the coordinates of the centroid G with respect to O and *x*,*y*,*z* be the coordinates of dm with respect to O. If x', y', z' are coordinates of dm with respect to G

$$x = x' + \bar{x}\ , y = y' + \bar{y} \quad \text{and} \quad z = z' + \bar{z}$$

From equation I_x

$$I_X = \int (y^2 + z^2)dm = \int [(y' + \bar{y})^2 + (z' + \bar{z})^2]\,dm$$

$$\therefore\ I_X = \int (y'^2 + z'^2)dm + 2\bar{y}\int y'dm + 2\bar{z}\int z'dm + (\bar{y}^2 + \bar{z}^2)\int dm$$

$$\int (y'^2 + z'^2)dm = I_{X'}$$

$\int y'dm = 0$ and $\int z'dm = 0$ as they represent the moment of mass about centroidal axis.

$$\int dm = m$$

$$\therefore \quad I_X = I_{X'} + m(\bar{y}^2 + \bar{z}^2)$$

Similarly,

$$I_Y = I_{Y'} + m(\bar{x}^2 + \bar{z}^2)$$

and

$$I_Z = I_{Z'} + m(\bar{x}^2 + \bar{y}^2)$$

I_x, I_y and I_z are moments of inertia about centroidal axes. $\bar{y}^2 + \bar{z}^2$ is the perpendicular distance between the centroidal X' axis and the X axis. Similarly, $\bar{x}^2 + \bar{z}^2$ is the perpendicular distance between Y' and Y axes, and, $\bar{x}^2 + \bar{y}^2$ is the perpendicular distance between Z' and Z axes.

• **The mass moment of inertia of a body about any axis is equal to the sum of the mass moment of inertia about a parallel centroidal axis and the product of mass and square of the distance between the two parallel axes.**

A general equation for the above theorem can be written as

$$I = I_G + md^2$$

where

I = Moment of inertia about a given axis,

I_G = Moment of inertia about a parallel centroidal axis.

and

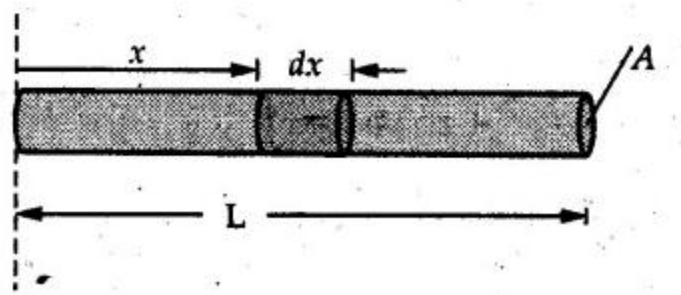
d = Distance between the two parallel axes

Solved Example

Determine the mass moment of inertia of a rod of length L and a small area of cross section A about an axis perpendicular to its length at its end.

Solution:

Consider a small element of length dx at distance x from one end as shown in Fig.



Volume of the element $dV = A\,dx$

Mass of the element dm = $\rho\, A\, dx$

$$dI = x^2\,dm = \rho A x^2\,dx$$

$$I = \int_0^L \rho A x^2\,dx = \frac{\rho A L^3}{3}$$

The mass of the rod is $m = \rho A L$

$$\therefore \quad \boxed{I = \frac{mL^2}{3}}$$