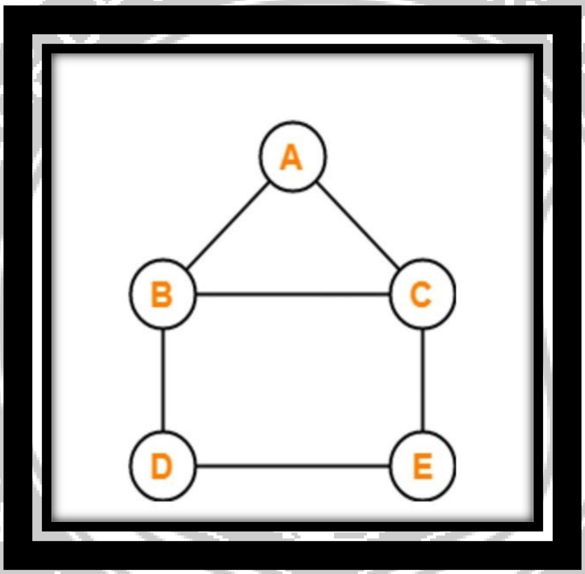
Paths, Reachability and Connectedness:

A path is a graph is a sequence $v_1, v_2, v_3, \ldots, v_k$ of vertices each adjacent to the next. In other words, starting with the vertex v_1, one can travel along edges $(v_1, v_2), (v_2, v_3)$, ... and reach the vertex v_k.



Length of the path:

The number of edges appearing in the sequence of a path is called the length of path.

Cycle or Circuit:

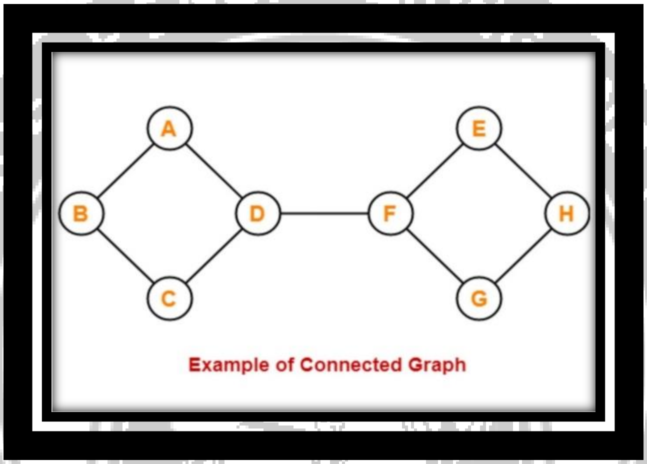
A path which originates and ends in the same node is called a cycle of circuit.

A path is said to be simple if all the edges in the path are distinct.

A path in which all the vertices are traversed only once is called an elementary path.

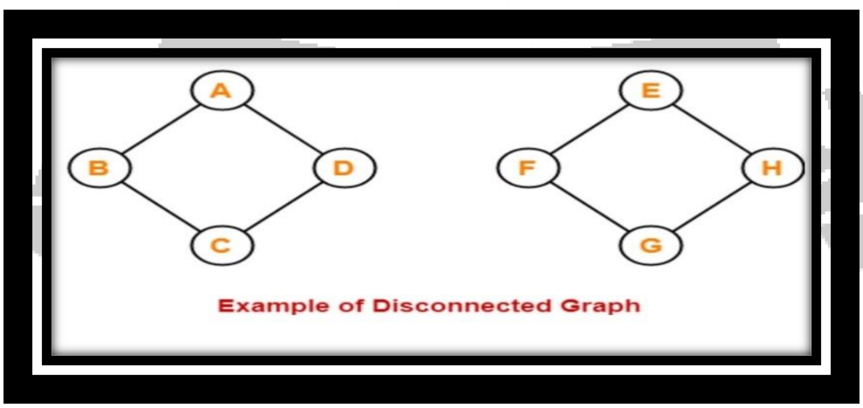
Connected Graph:

An directed graph is said to be connected if any pair of nodes are reachable from one another. That is, there is a path between any pair of nodes.



Disconnected graph:

A graph which is not connected is called disconnected graph.



Theorem: 1

If a graph has n vertices and a vertex v is connected to a vertex w, then there exists a path from v to w of length not more than $(n-1)$.

Proof:

Let $v, u_1, u_2, \ldots, u_{m-1}, w$ be a path in G from v to w.

By definition pf path, the vertices $v, u_1, u_2, \ldots, u_{m-1}$ and w all are distinct.

As G, contains only "n" vertices, it follows that $m + 1 \leq n$

$$\Rightarrow m \leq n - 1$$

Hence the proof.

Theorem: 2

Prove that a simple graph with n vertices must be connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

Proof:

Let G be a simple graph with n vertices and more than $\frac{(n-1)(n-2)}{2}$ edges.

Suppose if G is not connected, then G must have atleast two components. Let it be $G_1$and G_2.

Let V_1 be the vertex set of G_1 with $|V_1| = m$. If V_2 is the vertex set of G_2, then

$|V_2| = n - m$.

Then (i) $1 \leq m \leq n - 1$

(ii) There is no edge joining a vertex of V_1 and a vertex of V_2.

(iii) $|V_2| = n - m \geq 1$

Now, $|E(G)| = |E(G_1 \cup G_2)|$

$$= |E(G_1)| + |E(G_2)|$$

$$\leq \frac{m(m-1)}{2} + \frac{(n-m)(n-m-1)}{2}$$

$$= \frac{1}{2}[m^2 - m + n(n - m - 1) - m(n - m - 1)]$$

$$= \frac{1}{2}[n(n-1) - nm - m(n - m - 1) + m^2 - m]$$

$$= \frac{1}{2}[(n-1)(n-2) + 2(n-1) - 2nm + m^2 + m + m^2 - m]$$

Adding and Subtracting $2n - 2$

$$= \frac{1}{2}[(n-1)(n-2) + 2n - 2 - 2nm + 2m^2]$$

$$= \frac{1}{2}[(n-1)(n-2) + 2n(1-m) + 2(m^2 - 1)]$$

$$= \frac{1}{2}[(n-1)(n-2) - 2n(m-1) + 2(m-1)(m+1)]$$

$$= \frac{1}{2}[(n-1)(n-2) - 2(m-1)(n-m-1)]$$

$|E(G)| \leq \frac{(n-1)(n-2)}{2}$, Since $(m-1)(n-m-1) \geq 0$ for $1 \leq m \leq n-1$

Which is a contradiction as G has more than $\frac{(n-1)(n-2)}{2}$ edges.

Hence G is a connected graph.

Hence the proof.

Theorem: 3

Let G be a simple graph with n vertices. Show that if $\delta(G) \geq \left[\frac{n}{2}\right]$, then G is connected where $\delta(G)$ is minimum degree of the graph G.

Proof:

Let u and v be any two distinct vertices in the graph G.

We claim that there is a $u - v$ path in G.

Suppose uv is not an edge of G. Then, X be the set of all vertices which are adjacent to u and Y be the set of all vertices which are adjacent to v.

Then $u, v \notin X \cup Y$. (Since G is a simple graph)

And hence $|X \cup Y| \leq n - 2$

We have $|X| = deg(u) \geq \delta(G) \geq \left[\frac{n}{2}\right]$ and $|Y| = deg(v) \geq \delta(G) \geq \left[\frac{n}{2}\right]$

Now, $|X| + |Y| \geq \left[\frac{n}{2}\right] + \left[\frac{n}{2}\right] = n \geq n - 1$

We know that $|X \cup Y| = |X| + |Y| - |X \cap Y|$

$n - 2 \geq |X \cup Y| \geq n - 1 - |X \cap Y|$

We have, $|X \cap Y| \geq 1 \Rightarrow X \cap Y \neq \emptyset$

Now, take a vertex $w \in X \cap Y$. Then uvw is a $u - v$ path in G.

Thus for every pair of distinct vertices of G there is a path between them.

Hence G is connected.

Hence the proof.