1.1 The Principle of Inclusion – Exclusion:

Formula

$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| + |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$

$$|A_1 \cup A_2 \cup A_3 \cup A_4|$$
$$= |A_1| + |A_2| + |A_3| + |A_4| + |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|$$
$$- |A_1 \cap A_4| - |A_2 \cap A_4| - |A_3 \cap A_4| + |A_1 \cap A_2 \cap A_3|$$
$$+ |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_3 \cap A_4|$$
$$- |A_1 \cap A_2 \cap A_3 \cap A_4|$$

Problems under Inclusion and Exclusion:

1. How many positive integers not exceeding 1000 are divisible by 7 or 11?

Solution:

Let A denote the set of positive integers not exceeding 1000 that are divisible by 7.

Let B denote the set of positive integers not exceeding 1000 that are divisible by 11.

Then, $|A| = \left[\frac{1000}{7}\right] = [142.8] = 142$

$|B| = \left[\frac{1000}{11}\right] = [90.9] = 90$

$|A \cap B| = \left[\frac{1000}{7\times 11}\right] = [12.9] = 12$

The number of positive integer not exceeding 1000 that are divisible either 7 or 77 is $|A \cup B|$

By principle of inclusion – exclusion,

$|A \cup B| = |A| + |B| - |A \cap B|$

$= 142 + 90 - 12 = 220$

There are 220 positive integer not exceeding 1000 divisible by either 7 or 11.

2. Determine n such that $1 \leq n \leq 100$ which are not divisible by 5 or by 7.

Solution:

Let A denote the number n, $1 \leq n \leq 100$ which is divisible by 5.

Let B denote the number n, $1 \leq n \leq 100$ which is divisible by 7.

Then, $|A| = \left[\frac{100}{5}\right] = [20] = 20$

$|B| = \left[\frac{100}{7}\right] = [14.3] = 14$

$|A \cap B| = \left[\frac{1000}{5\times 7}\right] = [2.8] = 2$

Now, the number n, $1 \leq n \leq 100$ which is divisible by either 5 or 7 is $|A \cup B|$

By principle of inclusion – exclusion,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 20 + 14 - 2 = 32$$

The number $n,\ 1 \leq n \leq 100$ which is divisible by either 5 or 7 is

$$= 100 - 32 = 68$$

There are 68 number not exceeding 100 that are not divisible by either 5 or 7.

3. Find the number of integers between 1 to 250 that are not divisible by any of the integers 2, 3, 5 and 7

Solution:

Let A denote the integer from 1 to 250 that are divisible by 2.

Let B denote the integer from 1 to 250 that are divisible by 3.

Let C denote the integer from 1 to 250 that are divisible by 5.

Let D denote the integer from 1 to 250 that are divisible by 7.

Then, $|A| = \left[\frac{250}{2}\right] = 125$

$$|B| = \left[\frac{250}{3}\right] = 83$$

$$|C| = \left[\frac{250}{5}\right] = 50$$

$|D| = \left\lfloor \frac{250}{7} \right\rfloor = 35$

The number of integer between 1 – 250 that are divisible by 2 & 3

$|A \cap B| = \left\lfloor \frac{250}{2\times 3} \right\rfloor = 41$

$|A \cap C| = \left\lfloor \frac{250}{2\times 5} \right\rfloor = 25$

$|A \cap D| = \left\lfloor \frac{250}{2\times 7} \right\rfloor = 17$

$|B \cap C| = \left\lfloor \frac{250}{3\times 5} \right\rfloor = 16$

$|B \cap D| = \left\lfloor \frac{250}{3\times 7} \right\rfloor = 11$

$|C \cap D| = \left\lfloor \frac{250}{5\times 7} \right\rfloor = 7$

The number of integer between 1 – 250 that are divisible by 2, 3 & 5

$|A \cap B \cap C| = \left\lfloor \frac{250}{2\times 3\times 5} \right\rfloor = 8$

$|A \cap B \cap D| = \left\lfloor \frac{250}{2\times 3\times 7} \right\rfloor = 5$

$|A \cap C \cap D| = \left\lfloor \frac{250}{2\times 5\times 7} \right\rfloor = 3$

$|B \cap C \cap D| = \left\lfloor \frac{250}{3\times 5\times 7} \right\rfloor = 2$

$|A \cap B \cap C \cap D| = \left[\frac{250}{2\times3\times5\times7}\right] = 1$

The number of integer between 1 – 250 that are divisible by 2, 3, 5 & 7 is

$|A \cap B \cap C \cap D|$

By principle of inclusion and exclusion,

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

$$= (125 + 83 + 50 + 35) - (41 + 25 + 17 + 16 + 11 + 7) + (8 + 5 + 3 + 2) - 1$$

$$= 293 - 117 + 18 - 1 = 193$$

Now, the number of integer not divisible by any of 2, 3, 5 and 7

$$= 250 - 193 = 57$$

4. How many integers between 1 to 100 that are (i) not divisible by 7, 11 or 13 (ii) divisible by 3 but not by 7.

Solution:

Let A, B and C denote respectively the number of integer between 1 to 100 that are divisible by 7, 11 and 13 respectively.

Then, $|A| = \left[\frac{100}{7}\right] = 14$

$|B| = \left[\frac{100}{11}\right] = 9$

$|C| = \left[\frac{100}{13}\right] = 7$

$|A \cap B| = \left[\frac{100}{7\times11}\right] = 1$

$|A \cap C| = \left[\frac{100}{7\times13}\right] = 1$

$|B \cap C| = \left[\frac{100}{11\times13}\right] = 0$

$|A \cap B \cap C| = \left[\frac{100}{7\times11\times13}\right] = 0$

The number of integers between 1 – 100 that are divisible by 7, 11 and 13 is

$|A \cup B \cup C \cup D|$

By principle of inclusion and exclusion,

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| - |B \cap D| +$

$|A \cap B \cap C| - |A \cap B \cap C|$

$= (14 + 9 + 7) - (1 + 1 + 0) + (0)$

$= 30 - 2 = 28$

Now, the number of integer not divisible by any of 7, 11 and 13

$$= 100 - 28 = 72$$

Let A and B denote the number between 1 – 100that are divisible by 3 and 7 respectively.

Then, $|A| = \left[\frac{100}{3}\right] = 33$

$|B| = \left[\frac{100}{7}\right] = 14$

$|A \cap B| = \left[\frac{100}{3\times7}\right] = 4$

The number of integers divisible by 3 but not by 7.

$$= |A| - |A \cap B| = 33 - 4 = 29$$

5. Find the number of integers between 1 to 100 that are divisible by (i) 2, 3, 5 and 7 (ii) 2, 3, 5 but not by 7

Solution:

Let A, B, C and D denote the number of positive integers between 1 to 100 that are divisible by 2, 3, 5 and 7 respectively.

Then, $|A| = \left[\frac{100}{2}\right] = 50$

$$|B| = \left[\frac{100}{3}\right] = 33$$

$$|C| = \left[\frac{100}{5}\right] = 20$$

$$|D| = \left[\frac{100}{7}\right] = 14$$

$$|A \cap B| = \left[\frac{100}{2\times3}\right] = 16$$

$$|A \cap C| = \left[\frac{100}{2\times5}\right] = 10$$

$$|A \cap D| = \left[\frac{100}{2\times7}\right] = 7$$

$$|B \cap C| = \left[\frac{100}{3\times5}\right] = 6$$

$$|B \cap D| = \left[\frac{100}{3\times7}\right] = 4$$

$$|C \cap D| = \left[\frac{100}{5\times7}\right] = 2$$

$$|A \cap B \cap C| = \left[\frac{100}{7\times11\times13}\right] = 3$$

$$|A \cap B \cap D| = \left[\frac{100}{2\times3\times7}\right] = 2$$

$$|A \cap C \cap D| = \left[\frac{100}{2\times5\times7}\right] = 1$$

$$|B \cap C \cap D| = \left[\frac{100}{3\times5\times7}\right] = 0$$

$|A \cap B \cap C \cap D| = \left[\frac{100}{2\times3\times7\times11}\right] = 0$

By principle of inclusion and exclusion,

$$|A \cup B \cup C \cup D|$$
$$= |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| - |B \cap C|$$
$$- |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D|$$
$$+ |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

$$= (50 + 33 + 20 + 14) - (16 + 10 + 7 + 6 + 4 + 2) + (3 + 2 + 1 + 0) - 0$$

$$= 117 - 45 + 6 = 78$$

(ii) The number of integers between 1 – 100 that are divisible by 2, 3, 5 but not by 7 $= |A \cap B \cap C| - |A \cap B \cap C \cap D|$

$$= 3 - 0 = 3$$

6. Determine the number of positive integers n, $1 \leq n \leq 1000$, that are not divisible by 2, 3 or 5 but are divisible by 7.

Solution:

Let A, B, C and D denote the number of positive integers between 1 to 1000 that are divisible by 2, 3, 5 and 7 respectively.

$|D| = \left[\frac{1000}{7}\right] = [142.8] = 142$

$|A \cap B \cap C \cap D| = \left[\frac{1000}{2\times3\times5\times7}\right] = [4.7] = 4$

The number between 1 – 1000 that are divisible by 7 but not divisible by 2, 3, 5 and 7 $= |D| - |A \cap B \cap C \cap D|$

$= 42 - 4 = 138$

7. In a survey of 100 students, it was found that 30 studied Mathematics, 54 studied Statistics, 25 studied Operation research, 1 studied all the three subjects. 20 studied Mathematics and Statistics, 3 studied Mathematics and Operation Research and 15 studied Statistics and Operations Research, (i) How many students studied none of these subjects?(ii) How many students studied only Mathematics.

Solution:

Let A denote the students who studied Mathematics.

Let B denote the students who studied Statistics.

Let B denote the students who studied Operations Research.

It is given that $|A| = 30, |B| = 54, \; |C| = 25, |A \cap B| = 20, |A \cap C| = 3,$

$|B \cap C| = 15, |A \cap B \cap C| = 1$

By principle of inclusion – exclusion, the number of students playing either volleyball or hockey is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= 30 + 54 + 25 - 20 - 3 - 15 + 1$$

Students who studied none of these 3 subjects $= 100 - 72 = 28$

The number of students only studied Mathematics and Statistics $= 20 - 1 = 19$

The number of students only studied Mathematics and Operations Research

$= 3 - 1 = 2$

The number of students only studied Mathematics $= 30 - 19 - 2 - 1 = 8$

8. A survey of 500 from a school produced the following information. 200 play volleyball, 120 play hockey, 60 play both volleyball and hockey. How many are not playing either volleyball or hockey?

Solution:

Let A denote the students who play volleyball.

Let B denote the students who play hockey.

It is given that $n = 500, |A| = 200, |B| = 120, |A \cap B| = 60$

By principle of inclusion – exclusion, the number of students playing either volleyball or hockey is

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$= 200 + 120 - 60 = 260$$

The number of students not playing either volleyball or hockey is

$$= 500 - 260 = 240$$

9. Out of 100 students in a college, 38 play tennis, 57 play cricket and 31 play hockey, 9 play cricket and hockey, 10 play hockey and tennis, 12 play tennis and cricket. How many play (i) all three games (ii) just one game (iii) tennis and cricket but not hockey. (Assume that each student plays atleast one game)

Solution:

Let T, C and H denote the set of students playing Tennis, Cricket and Hockey respectively.

Given that $|T| = 38, |C| = 57, \; |H| = 31, |T \cap C| = 12, |T \cap H| = 10,$

$|C \cap H| = 9, |T \cup C \cup H| = 100$

Now, the number of integer who play all three games $= |T \cap C \cap H|$

By principle of inclusion – exclusion,

$$|T \cup C \cup H| = |T| + |C| + |H| - |T \cap C| - |T \cap H| - |C \cap H| + |T \cap C \cap H|$$

$$100 = 38 + 5 + 31 - 103 - 12 - 10 + 9 + |T \cap C \cap H|$$

$$|T \cap C \cap H| = 100 - 126 + 31 = 5$$

Number of students who play all 3 games = 5

Number of students playing just one game = number of students Tennis only + number of students playing cricket only + number of students playing Hockey only

$$= 21 + 41 + 17 = 79$$

The number of students playing Tennis and Cricket but not Hockey

$$= |T \cap C| - |T \cap C \cap H|$$

$$= 12 - 5 = 7$$

10. A survey of 500 television watchers produced the following information. 285 watch Hockey games. 195 watch Football games. 115 watch basketball games.70 watch football and hockey games. 50 watch hockey and basketball games and 30 watch football and basketball games. 50 do not watch any of the three games. How many people watch exactly one of the three games.

Solution:

Let H denotes the television watchers who watch Hockey.

Let F denotes the television watchers who watch Football.

Let B denotes the television watchers who watch Basket Ball.

Given that $|H| = 285, |F| = 195, |B| = 115, |H \cap F| = 70, |H \cap B| = 50,$

$|F \cap B| = 30$

Let x be the number of television watchers who watch all three games.

Now, we have

Given 50 members does not watch any of the three games.

Hence, $(165 + x) + (95 + x) + (35 + x) + (70 - x) + (50 - x) + (30 - x) +$

$x = 500$

$\Rightarrow 445 + x = 500$

$\Rightarrow x = 55$

Number of students who watches exactly one game is

$= 165 + x + 95 + x + 35 + x$

$= 295 + 3 \times 55 = 295 + 165 = 460$

11. A total of 1232 have taken a course in Tamil, 879 have taken a course in Telugu, and 114 have taken a course in Hindi. Further 103 have taken a course in both Tamil and Telugu, 23 have taken a course in Tamil and Hindi, and 14 have taken a course in Telugu and Hindi. If 2092 students have taken

atleast one of the Tamil, Telugu and Hindi, how many students have taken a course in all three languages.

Solution:

Let A denote the students who have taken a course in Tamil.

Let B denote the students who have taken a course in Telugu.

Let C denote the students who have taken a course in Hindi.

It is given that $|A| = 1232, |B| = 879, |C| = 114, |A \cap B| = 103, |A \cap C| = 23,$
$|B \cap C| = 14, |A \cup B \cup C| = 2092$

By principle of inclusion – exclusion, the number of students playing either volleyball or hockey is

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$2092 = 1232 + 879 + 114 - 103 - 23 - 14 + |A \cap B \cap C|$$

$$|A \cap B \cap C| = 2232 - 2225 = 7$$

Therefore, there are 7 students who have taken the course in Tamil, Telugu and Hindi.