3.3 INVERSE LAPLACE TRANSFORM

Definition

If the Laplace transform of a function $f(t)$ is $F(s) ie., L[f(t)] = F(s)$,then$f(t)$ is called an inverse Laplace transform of $F(s)$ and we write symbolically$f(t) = L^{-1}[F(s)]$, where L^{-1} is called the inverse Laplace transform operator.

Inverse Laplace transform of elementary functions

$L[f(t)] = F(s)$	$L^{-1}[F(s)] = f(t)$
$L[1] = \frac{1}{s}$	$L^{-1}\left[\frac{1}{s}\right] = 1$
$L[t] = \frac{1}{s^2}$	$L^{-1}\left[\frac{1}{s^2}\right] = t$
$L[t^n] = \frac{n!}{s^{n+1}}$ *if n is an integer*	$L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$ $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!}$
$L[e^{at}] = \frac{1}{s-a}$	$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
$L[e^{-at}] = \frac{1}{s+a}$	$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$
$L[sinat] = \frac{a}{s^2+a^2}$	$L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{sinat}{a}$
$L[cosat] = \frac{s}{s^2+a^2}$	$L^{-1}\left[\frac{s}{s^2+a^2}\right] = cosat$

$\boldsymbol{L[sinhat] = \frac{a}{s^2 - a^2}}$	$L^{-1}\left[\frac{\mathbf{1}}{\boldsymbol{s^2 - a^2}}\right] = \boldsymbol{\frac{sinhat}{a}}$
$\boldsymbol{L[cosat] = \frac{s}{s^2 - a^2}}$	$L^{-1}\left[\frac{\boldsymbol{s}}{\boldsymbol{s^2 - a^2}}\right] = \boldsymbol{coshat}$

Result on inverse Laplace transform

Result: 1 Linear property

$\boldsymbol{L[f(t)] = F(s)}$ **and** $\boldsymbol{L[g(t)] = G(s)}$ **,then** $\boldsymbol{L^{-1}[aF(s) \pm bG(s)] = aL^{-1}[F(s)] \pm bL^{-1}[G(s)]}$

Where a and b are constants.

Proof:

We know that $L[aF(s) \pm bG(s)] = aL[F(s)] \pm bL[G(s)]$

$$= a\,F(s) \pm b\,G(s)$$

$(i.e.)\, a\,F(s) \pm b\,G(s) = L[af(t) \pm bg(t)]$

Operating L^{-1} on both sides, we get

$L^{-1}[aF(s) \pm bG(s)] = af(t) \pm bg(t)$

$\because f(t) = L^{-1}[F(s)]$

$\because g(t) = L^{-1}[G(s)]$

$L^{-1}[aF(s) \pm bG(s)] = aL^{-1}[F(s)] \pm bL^{-1}[G(s)]$

Result: 2 First shifting property

(i) $\boldsymbol{L^{-1}\left[F(s+a) = e^{-at}L^{-1}[F(s)]\right]}$

(ii) $\boldsymbol{L^{-1}\left[F(s-a) = e^{at}L^{-1}[F(s)]\right]}$

Proof:

Let $L[e^{-at}f(t)] = F[s+a]$

Operating L^{-1} on both sides, we get

$e^{-at}f(t) = L^{-1}\big[F[s+a]\big]$

$L^{-1}\big[F[s+a]\big] = e^{-at}L^{-1}[F(s)]$

Result: 3 Multiplication by s.

If $\boldsymbol{L^{-1}[F(s)] = f(t)}$ **and** $\boldsymbol{f(0) = 0}$**, then** $\boldsymbol{L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]}$

Proof:

We know that $L[f'(t)] = sL[f(t)] - f(0) = sF(s)$

Operating L^{-1} on both sides, we get

$f'(t) = L^{-1}[sF(s)]$

$\frac{d}{dt}f(t) = L^{-1}[sF(s)]$

$\frac{d}{dt}L^{-1}[F(s)] = L^{-1}[sF(s)]$

$\therefore L^{-1}[sF(s)] = \frac{d}{dt}L^{-1}[F(s)]$

Result: 4 Division by s.

$\boldsymbol{L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t} L^{-1}[F(s)]dt$

Proof:

We know that $L\left[\int_0^t f(t)dt\right] = \frac{1}{s}L[f(t)] = \frac{1}{s}F(s)$

Operating L^{-1} on both sides ,we get

$\int_0^t f(t)dt = L^{-1}\left[\frac{1}{s}F(s)\right]$

$\int_0^t L^{-1}[F(s)]\,dt = L^{-1}\left[\frac{1}{s}F(s)\right]$

$\therefore L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]\,dt$

Result: 5 Inverse Laplace transform of derivative

$\boldsymbol{L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]}$

Proof:

We know that $L[tf(t)] = \frac{-d}{ds}L[f(t)] = \frac{-d}{ds}F(s)$

Operating L^{-1} on both sides ,we get

$tf(t) = -L^{-1}\left[\frac{d}{ds}F(s)\right]$

$L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$

$f(t) = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$

$L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]$

Result: 6 Inverse Laplace transform of integral

$\boldsymbol{L^{-1}[F(s)] = t}L^{-1}\left[\int_s^\infty F(s)ds\right]$

Proof:

We know that $L\left[\frac{f(t)}{t}\right] = \int_s^\infty L(f(t))\,ds$

$$= \int_s^\infty F(s)\, ds$$

Operating L^{-1} on both sides, we get

$$\frac{f(t)}{t} = L^{-1}\left[\int_s^\infty F(s)\, ds\right]$$

$$f(t) = tL^{-1}\left[\int_s^\infty F(s)\, ds\right]$$

$$L^{-1}[F(s)] = tL^{-1}\left[\int_s^\infty F(s)\, ds\right]$$

Problems under inverse Laplace transform of elementary functions

Example: Find the inverse Laplace for the following

(i) $\frac{1}{2s+3}$ **(ii)** $\frac{1}{4s^2+9}$ **(iii)** $\frac{s^3-3s^2+7}{s^4}$ **(iv)** $\frac{3s+5}{s^2+36}$

Solution:

(i) $L^{-1}\left[\frac{1}{2s+3}\right] = L^{-1}\left[\frac{1}{2\left[s+\frac{3}{2}\right]}\right]$

$$= \frac{1}{2} e^{-\frac{-3t}{2}}$$

(ii) $L^{-1}\left[\frac{1}{4s^2+9}\right] = L^{-1}\left[\frac{1}{4\left[s^2+\frac{9}{4}\right]}\right]$

$$= \frac{1}{4} L^{-1}\left[\frac{1}{\left[s^2+\frac{9}{4}\right]}\right]$$

$$= \frac{1}{4}\frac{1}{3/2} sin \frac{3}{2} t$$

$$= \frac{1}{6} sin \frac{3}{2} t$$

(iii) $L^{-1}\left[\frac{s^3-3s^2+7}{s^4}\right] = L^{-1}\left[\frac{s^3}{s^4} - \frac{3s^2}{s^4} + \frac{7}{s^4}\right]$

$$= L^{-1}\left[\frac{1}{s}\right] - 3L^{-1}\left[\frac{1}{s^2}\right] + 7L^{-1}\left[\frac{1}{s^4}\right]$$

$$L^{-1}\left[\frac{s^3-3s^2+7}{s^4}\right] = 1 - 3t + \frac{7t^3}{3!}$$

(iv) $L^{-1}\left[\frac{3s+5}{s^2+36}\right] = 3L^{-1}\left[\frac{s}{s^2+36}\right] + 5L^{-1}\left[\frac{1}{s^2+36}\right]$

$$L^{-1}\left[\frac{3s+5}{s^2+36}\right] = 3cos6t + \frac{5sin6t}{6}$$

Inverse Laplace transform using First shifting theorem

$$\boldsymbol{L^{-1}[F(s+a)] = e^{-at}L^{-1}[F(s)]}$$

Example: Find the inverse Laplace transform for the following:

(i) $\frac{1}{(s+2)^2}$ **(ii)** $\frac{1}{(s-3)^4}$ **(iii)** $\frac{1}{(s+3)^2+9}$ **(iv)** $\frac{1}{s^2-2s+2}$

(v) $\frac{1}{s^2-4s+13}$ **(vi)** $\frac{s+2}{(s+2)^2+25}$ **(vii)** $\frac{s+2}{s^2+4s+20}$ **(viii)** $\frac{s}{(s+3)^2}$

(ix) $\frac{s}{(s-4)^3}$ **(x)** $\frac{s}{s^2-2s+2}$ **(xi)** $\frac{2s+3}{s^2+6s+25}$ **(xii)** $\frac{s}{s^2+6s-7}$

Solution:

(i) $L^{-1}\left[\frac{1}{(s+2)^2}\right] = e^{-2t}L^{-1}\left[\frac{1}{s^2}\right] = e^{-2t}t$

(ii) $L^{-1}\left[\frac{1}{(s-3)^4}\right] = e^{3t}L^{-1}\left[\frac{1}{s^4}\right] = e^{-2t}\frac{t^3}{3!}$

(iii) $L^{-1}\left[\frac{1}{(s+3)^2+9}\right] = e^{-3t}L^{-1}\left[\frac{1}{s^2+9}\right] = e^{-3t}\frac{sin3t}{3}$

(iv) $L^{-1}\left[\frac{1}{s^2-2s+2}\right] = L^{-1}\left[\frac{1}{(s-1)^2+1}\right] = e^{t}L^{-1}\left[\frac{1}{s^2+1}\right] = e^{t}sint$

(v) $L^{-1}\left[\frac{1}{s^2-4s+13}\right] = L^{-1}\left[\frac{1}{(s-2)^2+9}\right] = e^{2t}L^{-1}\left[\frac{1}{s^2+9}\right] = e^{2t}\frac{sin3t}{3}$

(vi) $L^{-1}\left[\frac{s+2}{(s+2)^2+25}\right] = e^{-2t}L^{-1}\left[\frac{s}{s^2+25}\right] = e^{-2t}cos5t$

(vii) $L^{-1}\left[\frac{s+2}{s^2+4s+20}\right] = L^{-1}\left[\frac{s+2}{(s+2)^2+16}\right]$

$= e^{-2t}L^{-1}\left[\frac{s}{s^2+16}\right] = e^{-2t}cos4t$

(viii) $L^{-1}\left[\frac{s}{(s+3)^2}\right] = L^{-1}\left[\frac{s+3-3}{(s+3)^2}\right]$

$= L^{-1}\left[\frac{s+3}{(s+3)^2}\right] - L^{-1}\left[\frac{3}{(s+3)^2}\right]$

$= L^{-1}\left[\frac{1}{s+3}\right] - 3L^{-1}\left[\frac{1}{(s+3)^2}\right]$

$= e^{-3t} - 3e^{-3t}L^{-1}\left[\frac{1}{s^2}\right]$

$= e^{-3t} - 3e^{-3t}t$

(ix) $L^{-1}\left[\frac{s}{(s-4)^3}\right] = L^{-1}\left[\frac{s-4+4}{(s-4)^3}\right]$

$= L^{-1}\left[\frac{s-4}{(s-4)^3}\right] + L^{-1}\left[\frac{4}{(s-4)^3}\right]$

$= L^{-1}\left[\frac{1}{(s-4)^2}\right] + 4L^{-1}\left[\frac{1}{(s-4)^3}\right]$

$= e^{4t}L^{-1}\left[\frac{1}{s^2}\right] + 4e^{4t}L^{-1}\left[\frac{1}{s^3}\right]$

$= e^{4t}t + 4e^{4t}\frac{t^2}{2!}$

$= e^{4t}t + 2e^{4t}t^2$

(x) $L^{-1}\left[\frac{s}{s^2-2s+2}\right] = L^{-1}\left[\frac{s}{(s-1)^2+1}\right] = L^{-1}\left[\frac{s-1+1}{(s-1)^2+1}\right]$

$= L^{-1}\left[\frac{s-1}{(s-1)^2+1}\right] + L^{-1}\left[\frac{1}{(s-1)^2+1}\right]$

$= e^{t}L^{-1}\left[\frac{s}{s^2+1}\right] + e^{t}L^{-1}\left[\frac{1}{s^2+1}\right]$

$$L^{-1}\left[\frac{s}{s^2-2s+2}\right] = e^t cost + e^t sint$$

(xi) $$L^{-1}\left[\frac{2s+3}{s^2+6s+25}\right] = L^{-1}\left[\frac{2s+3}{(s+3)^2+16}\right] = L^{-1}\left[\frac{2(s+3-3)+3}{(s+3)^2+16}\right]$$

$$= L^{-1}\left[\frac{2(s+3)-6+3}{(s+3)^2+16}\right]$$

$$= e^{-3t}L^{-1}\left[\frac{2s-3}{s^2+16}\right]$$

$$= e^{-3t}\left[2L^{-1}\left[\frac{s}{s^2+16}\right] - 3L^{-1}\left[\frac{1}{s^2+16}\right]\right]$$

$$L^{-1}\left[\frac{2s+3}{s^2+6s+25}\right] = e^{-3t}\left(2cos4t - \frac{3sin4t}{4}\right)$$

(xii) $$L^{-1}\left[\frac{s}{s^2+6s-7}\right] = L^{-1}\left[\frac{s}{(s+3)^2-16}\right] = L^{-1}\left[\frac{s+3-3}{(s+3)^2-16}\right]$$

$$= e^{-3t}L^{-1}\left[\frac{s-3}{s^2-16}\right]$$

$$= e^{-3t}L^{-1}\left[\frac{s}{s^2-16}\right] - 3e^{-3t}L^{-1}\left[\frac{1}{s^2-16}\right]$$

$$L^{-1}\left[\frac{s}{s^2+6s-7}\right] = e^{-3t}\left[cosh4t - \frac{3sinh4t}{4}\right]$$

Inverse using the formula

$$\boldsymbol{L^{-1}[F(s)] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}F(s)\right]}$$

Note: This formula is used when $F(s)$ is $\cot^{-1}\emptyset(s)$ or $\tan^{-1}\emptyset(s)$ or $log\emptyset(s)$

Example: Find the inverse Laplace transform for the following

(i) $\boldsymbol{cot^{-1}\left(\frac{s}{a}\right)}$ **(ii)** $\boldsymbol{tan^{-1}\left(\frac{a}{s}\right)}$ **(iii)** $\boldsymbol{cot^{-1}as}$

(iv) $\boldsymbol{tan^{-1}(s+a)}$ **(v)** $\boldsymbol{log\left(\frac{s+a}{s+b}\right)}$ **(vi)** $\boldsymbol{cot^{-1}\left(\frac{2}{s+1}\right)}$ **(vii)** $\boldsymbol{tan^{-1}\left(\frac{2}{s^2}\right)}$

Solution:

(i) $$L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(cot^{-1}\left(\frac{s}{a}\right)\right)\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{-1}{1+\frac{s^2}{a^2}}\left(\frac{1}{a}\right)\right] = \frac{1}{t}L^{-1}\left[\frac{-1}{\frac{a^2+s^2}{a^2}}\left(\frac{1}{a}\right)\right]$$

$$= \frac{1}{t}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right] = \frac{1}{t}sinat$$

(ii) $$L^{-1}\left[tan^{-1}\left(\frac{a}{s}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(tan^{-1}\left(\frac{a}{s}\right)\right)\right]$$

$$= \frac{-1}{t}L^{-1}\left[\frac{1}{1+\left(\frac{a}{s}\right)^2}\left(\frac{-a}{s^2}\right)\right] = \frac{-1}{t}L^{-1}\left[\frac{1}{\frac{s^2+a^2}{s^2}}\left(\frac{-a}{s^2}\right)\right]$$

$$= \frac{1}{t}L^{-1}\left[\frac{a}{s^2+a^2}\right]$$

$$L^{-1}\left[tan^{-1}\left(\frac{a}{s}\right)\right]=\frac{1}{t}sinat$$

(iii) $L^{-1}[cot^{-1}as]=\frac{-1}{t}L^{-1}\left[\frac{d}{ds}(cot^{-1}(as))\right]$

$$=\frac{-1}{t}L^{-1}\left[\frac{-1}{1+a^2s^2}(a)\right]=\frac{1}{t}L^{-1}\left[\frac{a}{a^2\left(s^2+\frac{1}{a^2}\right)}\right]$$

$$=\frac{1}{at}L^{-1}\left[\frac{1}{s^2+\frac{1}{a^2}}\right]=\frac{1}{at}\left[\frac{sin\frac{1}{a}t}{\frac{1}{a}}\right]$$

$$L^{-1}[cot^{-1}as]=\frac{1}{t}sin\frac{t}{a}$$

(iv) $L^{-1}[tan^{-1}(s+a)]=e^{-at}L^{-1}[tan^{-1}s]$

$$=e^{-at}\left[\frac{-1}{t}L^{-1}\left[\frac{d}{ds}(tan^{-1}s)\right]\right]$$

$$=e^{-at}\left(\frac{-1}{t}\right)L^{-1}\left[\frac{1}{1+s^2}\right]$$

$$=\frac{-1}{t}e^{-at}L^{-1}\left[\frac{1}{1+s^2}\right]$$

$$L^{-1}\left[cot^{-1}\left(\frac{s}{a}\right)\right]=\frac{-e^{-at}}{t}sint$$

(v) $L^{-1}\left[log\left(\frac{s+a}{s+b}\right)\right]=\frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(log\left(\frac{s+a}{s+b}\right)\right)\right]$

$$=\frac{-1}{t}L^{-1}\left[\frac{d}{ds}(\log(s+a)-\log(s+b))\right]$$

$$=\frac{-1}{t}L^{-1}\left[\frac{1}{s+a}-\frac{1}{s+b}\right]$$

$$=\frac{-1}{t}[e^{-at}-e^{-bt}]$$

$$L^{-1}\left[log\left(\frac{s+a}{s+b}\right)\right]=\frac{-1}{t}[e^{-at}-e^{-bt}]$$

(vi) $L^{-1}\left[cot^{-1}\left(\frac{2}{s+1}\right)\right]=e^{-t}L^{-1}\left[cot^{-1}\left(\frac{2}{s}\right)\right]$

$$=e^{-t}\left(\frac{-1}{t}\right)L^{-1}\left[\frac{d}{ds}\left(cot^{-1}\left(\frac{2}{s}\right)\right)\right]$$

$$=e^{-t}\left(\frac{-1}{t}\right)L^{-1}\left[\frac{-1}{1+\frac{4}{s^2}}\left(\frac{-2}{s^2}\right)\right]=-\frac{e^{-t}}{t}L^{-1}\left[\frac{1}{\frac{s^2+4}{s^2}}\left(\frac{2}{s^2}\right)\right]$$

$$=-\frac{e^{-t}}{t}L^{-1}\left[\frac{2}{s^2+4}\right]$$

$$L^{-1}\left[cot^{-1}\left(\frac{2}{s+1}\right)\right]=-\frac{e^{-t}}{t}sin2t$$

(vii) $L^{-1}\left[tan^{-1}\left(\frac{2}{s^2}\right)\right]=\frac{-1}{t}L^{-1}\left[\frac{d}{ds}\left(tan^{-1}\left(\frac{2}{s^2}\right)\right)\right]$

$$=\frac{-1}{t}L^{-1}\left[\frac{1}{1+\left(\frac{2}{s^2}\right)^2}\left(\frac{-4}{s^3}\right)\right]=\frac{4}{t}L^{-1}\left[\frac{1}{\frac{s^4+4}{s^4}}\left(\frac{1}{s^3}\right)\right]$$

$$= \frac{4}{t} L^{-1}\left[\frac{s}{s^4+4}\right]$$

$$= \frac{4}{t} L^{-1}\left[\frac{s}{(s^2)^2+2^2}\right] =$$

$$\frac{4}{t} L^{-1}\left[\frac{s}{(s^2+2)^2-(2s)^2}\right]$$

$$\because a^2 + b^2 = (a+b)^2 - 2ab$$

$$= \frac{4}{t} L^{-1}\left[\frac{s}{(s^2+2+2s)(s^2+2-2s)}\right]$$

$$= \frac{4}{t} L^{-1}\left[\frac{s}{-4s}\left(\frac{1}{s^2+2+2s} - \frac{1}{s^2+2-2s}\right)\right] \quad \because \left\{\begin{array}{c} \frac{1}{(s^2+ax+b)(s^2+ax+c)} \\ = \frac{1}{c-b}\left[\frac{1}{s^2+ax+b} - \frac{1}{s^2+ax+c}\right] \end{array}\right\}$$

$$= \frac{-1}{t} L^{-1}\left[\left(\frac{1}{s^2+2s+2} - \frac{1}{s^2-2s+2}\right)\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{1}{(s+1)^2+1} - \frac{1}{(s-1)^2+1}\right]$$

$$= \frac{-1}{t}\left(e^{-t} L^{-1}\left[\frac{1}{s^2+1}\right] - e^{t} L^{-1}\left[\frac{1}{s^2+1}\right]\right)$$

$$= \frac{-1}{t}(e^{-t} sint - e^{t} sint)$$

$$= \frac{sint}{t}(e^{-t} - e^{t})$$

$$= \frac{sint}{t} 2sinht$$

$$L^{-1}\left[tan^{-1}\left(\frac{2}{s^2}\right)\right] = \frac{2sintsinht}{t}$$

Inverse using the formula

$$\boldsymbol{L^{-1}[sF(s)] = \frac{d}{dt} L^{-1}[F(s)]}$$

Example: Find $\boldsymbol{L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]}$

Solution:

$$L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{d}{dt} L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] \cdots (1)$$

$$L^{-1}\left[log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = L^{-1}\frac{d}{ds}\left[log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{d}{ds}(\log(s^2+a^2) - \log(s^2+b^2))\right]$$

$$= \frac{-1}{t} L^{-1}\left[\frac{1}{s^2+a^2} 2s - \frac{1}{s^2+b^2} 2s\right]$$

$$= \frac{-2}{t} L^{-1}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]$$

$$= \frac{-2}{t}[cosat - cosbt]$$

$$= \frac{2}{t}[cosbt - cosat]$$

Substituting in (1), we get

$$L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = \frac{d}{dt}\left[\frac{2}{t}[cosbt - cosat]\right]$$

$$= 2\left[\frac{t(-bsinbt+asinat)-(cosbt-cosat)}{t^2}\right]$$

$$L^{-1}\left[slog\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = 2\left[\frac{t(-bsinbt+asinat)-(cosbt-cosat)}{t^2}\right]$$

Inverse using the formula

$$\boldsymbol{L^{-1}\left[\frac{F(s)}{s}\right] = \int_0^t L^{-1}[F(s)]dt}$$

This formula is used when $F(s) = \frac{one\ term}{s(another\ term)}$

Example: Find $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

Solution:

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s^2+a^2)}\right]dt$$

$$= \int_0^t \left[\frac{sinat}{a}\right]dt$$

$$= \frac{1}{a}\left[\frac{-cosat}{a}\right]_0^t$$

$$= \frac{-1}{a^2}[cosat]_0^t$$

$$= \frac{-1}{a^2}(cosat - cos0) = \frac{-1}{a^2}(cosat - 1)$$

$$\therefore L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1-cosat}{a^2}$$

Example: Find $L^{-1}\left[\frac{1}{s(s^2-a^2)}\right]$

Solution:

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s^2-a^2)}\right]dt$$

$$= \int_0^t \left[\frac{sinhat}{a}\right]dt$$

$$= \frac{1}{a}\left[\frac{coshat}{a}\right]_0^t$$

$$= \frac{1}{a^2}[coshat]_0^t$$

$$= \frac{1}{a^2}(coshat - cosh0) = \frac{1}{a^2}(coshat - 1)$$

$$\therefore L^{-1}\left[\frac{1}{s(s^2-a^2)}\right] = \frac{coshat-1}{a^2}$$

Example: Find $L^{-1}\left[\frac{1}{s(s+a)}\right]$

Solution:

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left[\frac{1}{(s+a)}\right] dt$$

$$= \int_0^t e^{-at} dt$$

$$= \left[\frac{e^{-at}}{-a}\right]_0^t$$

$$= \frac{-1}{a}(e^{-at} - 1)$$

$$\therefore L^{-1}\left[\frac{1}{s(s+a)}\right] = \frac{1-e^{-at}}{a}$$

Inverse using Partial Fraction

Example: Find $L^{-1}\left[\frac{s-2}{s(s+2)(s-1)}\right]$

Solution:

$$\frac{s-2}{s(s+2)(s-1)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s-1}$$

$$= \frac{A(s+2)(s-1)+Bs(s-1)+Cs(s+2)}{s(s+2)(s-1)}$$

$$A(s+2)(s-1) + Bs(s-1) + Cs(s+2) = s - 2 \cdots (1)$$

Put $s = 0$ in (1)	Put $s = -2$ in (1)	Put $s = 1$ in (1)
$A(2)(-1) = -2$	$B(-2)(-3) = -4$	$3C = -1$
$\Rightarrow A = 1$	$\Rightarrow B = \frac{-4}{6} = \frac{-2}{3}$	$\Rightarrow C = \frac{-1}{3}$

$$\therefore \frac{s-2}{s(s+2)(s-1)} = \frac{1}{s} - \frac{2}{s+2} - \frac{1}{3(s-1)}$$

$$L^{-1}\left[\frac{s-2}{s(s+2)(s-1)}\right] = L^{-1}\left[\frac{1}{s}\right] - \frac{2}{3}L^{-1}\left[\frac{1}{s+2}\right] - \frac{1}{3}L^{-1}\left[\frac{1}{s-1}\right]$$

$$L^{-1}\left[\frac{s-2}{s(s+2)(s-1)}\right] = 1 - \frac{2}{3}e^{-2t} - \frac{1}{3}e^{t}$$

Example: Find $L^{-1}\left[\frac{2s-3}{(s-1)(s-2)^2}\right]$

Solution:

$$\frac{2s-3}{(s-1)(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$$

$$= \frac{A(s-2)^2+B(s-1)(s-2)+C(s-1)}{(s-1)(s-2)^2}$$

$$A(s-2)^2 + B(s-1)(s-2) + C(s-1) = 2s - 3 \cdots (1)$$

Put $s = 1$ in (1)	Put $s = 2$ in (1)	Equating the coefficient of s^2
		$A + B = 0$
$A = -1$	$C = 1$	$B = -A \Rightarrow B = 1$

$$\therefore \frac{2s-3}{(s-1)(s-2)^2} = \frac{-1}{s-1} + \frac{1}{s-2} + \frac{1}{(s-2)^2}$$

$$L^{-1}\left[\frac{2s-3}{(s-1)(s-2)^2}\right] = -L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[\frac{1}{s-2}\right] + \left[\frac{1}{(s-2)^2}\right]$$

$$= -e^{t} + e^{2t} + e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$\therefore L^{-1}\left[\frac{2s-3}{(s-1)(s-2)^2}\right] = -e^{t} + e^{2t} + e^{2t}t$$

Example: Find the inverse Laplace transform of $\frac{5s^2-15s-11}{(s+1)(s-2)^3}$

Solution:

$$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{C}{(s-2)^3}$$

$$= \frac{A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)}{(s-1)(s-2)^3}$$

$$A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1) = 5s^2 - 15s - 11 \cdots (1)$$

Put $s = -1$ in (1)

$A(-27) = 9$

$A = \frac{9}{-27} \Rightarrow A = \frac{-1}{3}$

Put $s = 2$ in (1)

$D(3) = -21$

$D = \frac{-21}{3} = -7$

Equating the coefficient of s^3

$A + B = 0$

$B = -A \Rightarrow B = \frac{1}{3}$

Put $s = 0$ in (1), we get

$-8A + 4B - 2C + D = -11$

$\frac{8}{3} + \frac{4}{3} - 2C - 7 = -11$

$4 - 2C = 7 - 11$

$-2C = -8 \Rightarrow C = 4$

$$\therefore \frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{-1}{3(s+1)} + \frac{1}{3(s-2)} + \frac{4}{(s-2)^2} - \frac{7}{(s-2)^3}$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right] = \frac{-1}{3}L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s-2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2}\right] - 7L^{-1}\left[\frac{1}{(s-2)^3}\right]$$

$$= \frac{-1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}L^{-1}\left[\frac{1}{s^2}\right] - 7e^{2t}L^{-1}\left[\frac{1}{s^3}\right]$$

$$L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right] = \frac{-1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}L^{-1}\left[\frac{1}{s^2}\right] - 7e^{2t}\frac{t^2}{2}$$

Example: Find the inverse Laplace transform of $\frac{4s+5}{(s+1)(s^2+4)}$

Solution:

$$\frac{4s+5}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+c}{s^2+4}$$

$$= \frac{A(s^2+4)+(Bs+c)(s+1)}{(s+1)(s^2+4)}$$

$A(s^2+4)+(Bs+c)(s+1) = 4s+5 \cdots\cdots (1)$

Put $s = -1$in (1)	Equating coefficients of s^2term in (1)	Put $s = 0$in (1)
$A(1+4)+0 = 4(-1)+5$	$A+B=0$	$A(4)+C=5$
$A(5) = 1 \Rightarrow A = \frac{1}{5}$	$B = -A \Rightarrow B = \frac{-1}{5}$	$C = 5-4A =$

$5 - \frac{4}{5}$

$= \frac{25-4}{5} = \frac{21}{5}$

$$\therefore \frac{4s+5}{(s+1)(s^2+4)} = \frac{\frac{1}{5}}{s+1} + \frac{\frac{-1}{5}s+\frac{21}{5}}{s^2+4}$$

$$= \frac{1}{5(s+1)} - \frac{s}{5(s^2+4)} + \frac{21}{5}\frac{1}{(s^2+4)}$$

$$L^{-1}\left[\frac{4s+5}{(s+1)(s^2+4)}\right] = \frac{1}{5}L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{5}L^{-1}\left[\frac{s}{s^2+4}\right] + \frac{21}{5}L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}cos2t + \frac{21}{5}\frac{sin2t}{2}$$

$$L^{-1}\left[\frac{4s+5}{(s+1)(s^2+4)}\right] = \frac{1}{5}e^{-t} - \frac{1}{5}cos2t + \frac{21}{10}sin2t$$

3.3(a)INITIAL AND FINAL VALUE THEOREMS

Initial value theorem

Statement: If$L[f(t)] = F(s)$, then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Proof:

We know that $L[f'(t)] = s\,L[f(t)] - f(0)$

$$= sF(s) - f(0)$$

$$\therefore sF(s) = L[f'(t)] + f(0)$$

$$= \int_0^\infty e^{-st} f'(t)dt + f(0)$$

Taking limit as $s \to \infty$ on both sides, we have

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\int_0^\infty e^{-st} f'(t)dt + f(0)\right]$$

$$= \lim_{s\to\infty}\left[\int_0^\infty e^{-st} f'(t)dt\right] + f(0)$$

$$= \int_0^\infty \lim_{s\to\infty}[e^{-st} f'(t)]dt + f(0)$$

$$= 0 + f(0) \qquad \because e^{-\infty} = 0$$

$$= f(0)$$

$$= \lim_{t \to 0} f(t)$$

$$\therefore \lim_{s \to \infty} sF(s) = \lim_{t \to 0} f(t)$$

Final value theorem

Statement: If the Laplace transforms of $f(t)$ and $f'(t)$ exist and $L[f(t)] = F(s)$,then

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Proof:

We know that $L[f'(t)] = s\, L[f(t)] - f(0)$

$$= sF(s) - f(0)$$

$$\therefore sF(s) = L[f'(t)] + f(0)$$

$$= \int_0^\infty e^{-st} f'(t) dt + f(0)$$

Taking limit as $s \to 0$ on both sides, we have

$$\lim_{s \to 0} sF(s) = \lim_{s \to 0} \left[\int_0^\infty e^{-st} f'(t) dt + f(0)\right]$$

$$= \lim_{s \to 0} \left[\int_0^\infty e^{-st} f'(t) dt\right] + f(0)$$

$$= \int_0^\infty \lim_{s \to 0} [e^{-st} f'(t)] dt + f(0)$$

$$= \int_0^\infty f'(t) dt + f(0)$$

$$= [f(t)]_0^\infty + f(0)$$

$$= f(\infty) - f(0) + f(0)$$

$$= f(\infty)$$

$$= \lim_{t \to \infty} f(t)$$

$$\therefore \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Example: Verify the initial value theorem for the function $f(t) = ae^{-bt}$

Solution:

Given $f(t) = ae^{-bt}$

$$F(s) = L[f(t)]$$

$$= L[ae^{-bt}]$$

$$= a\, \frac{1}{s+b}$$

$$sF(s) = \frac{as}{s+b}$$

Initial value theorem is $\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0} ae^{-bt}$$

$$= a \cdots\cdots\cdots (1)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{as}{s+b}\right]$$

$$= \lim_{s\to\infty}\left[\frac{as}{s\left(1+\frac{b}{s}\right)}\right] == \lim_{s\to\infty}\left[\frac{a}{\left(1+\frac{b}{s}\right)}\right]$$

$$= a \cdots\cdots\cdots (2)$$

From (1) and (2), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

∴ Initial value theorem is verified

Example:Verify the initial value theorem and Final value theorem for the function $\boldsymbol{f(t) = 1 + e^{-t}[sint + cost]}$.

Solution:

Given $f(t) = 1 + e^{-t}[sint + cost]$

$$F(s) = L[f(t)]$$

$$= L\left[1 + e^{-t}[sint + cost]\right]$$

$$= L[1] + L\left[e^{-t}[sint + cost]\right]$$

$$= L[1] + L[sint + cost]_{s\to s+1}$$

$$= \frac{1}{s} + \left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]_{s\to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2+1} + \frac{s+1}{(s+1)^2+1}$$

$$F(s) = \frac{1}{s} + \frac{1}{s^2+2s+2} + \frac{s+1}{s^2+2s+2}$$

$$sF(s) = 1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2}$$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}\left[1 + e^{-t}[sint + cost]\right]$$

$$= 1 + 0 + 1 = 2 \cdots\cdots\cdots (1)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2}\right]$$

$$= 1 + \lim_{s\to\infty}\left[\frac{1}{s\left(1+\frac{2}{s}+\frac{2}{s^2}\right)} + \frac{\left(1+\frac{1}{s}\right)}{\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}\right]$$

$$= 1 + 0 + 1 = 2 \cdots\cdots\cdots (2)$$

From (1) and (2), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

∴ Initial value theorem is verified

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}(1 + e^{-t}[sint + cost])$

$= 1 + 0 = 1 \cdots\cdots\cdots (3)$

$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[1 + \frac{s}{s^2+2s+2} + \frac{s^2+s}{s^2+2s+2}\right]$

$= 1 + 0 + 0 = 1 \cdots\cdots\cdots (4)$

From (3) and (4), $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

∴ Final value theorem is verified.

Example: Verify the initial value theorem and Final value theorem for the function

$\boldsymbol{f(t) = L^{-1}\left[\frac{1}{s(s+2)^2}\right]}$

Solution:

Given $f(t) = L^{-1}\left[\frac{1}{s(s+2)^2}\right] \cdots (1)$

$= \int_0^t L^{-1}\left[\frac{1}{(s+2)^2}\right] dt = \int_0^t e^{-2t} L^{-1}\left[\frac{1}{s^2}\right] dt$

$= \int_0^t e^{-2t} t \, dt$

$= \int_0^t te^{-2t} \, dt$

$= \left[t\left(\frac{e^{-2t}}{-2}\right) - \frac{(1)e^{-2t}}{(-2)^2}\right]_0^t$

$= -t\frac{e^{-2t}}{2} - \frac{e^{-2t}}{4} - 0 + \frac{1}{4}$

$\therefore f(t) = \frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}$

From (1), $F(s) = \frac{1}{s(s+2)^2}$

$sF(s) = \frac{1}{(s+2)^2}$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$\lim_{t\to 0} f(t) = \lim_{t\to 0}\left[\frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}\right]$

$= \frac{1}{4} - 0 - \frac{1}{4} = 0$

$\therefore \lim_{t\to 0} f(t) = 0 \cdots (2)$

$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \frac{1}{(s+2)^2} = 0$

$\therefore \lim_{s\to\infty} sF(s) = 0 \cdots (3)$

From (2) and (3), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

∴ Initial value theorem is verified

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}\left[\frac{1}{4} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}\right]$$

$$= \frac{1}{4} - 0 - 0 = \frac{1}{4} \cdots (4)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{1}{4} \cdots (5)$$

From (4) and (5), $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

∴ Final value theorem is verified

Example: Verify the initial value theorem and Final value theorem for the function

$\boldsymbol{f(t) = e^{-t}(t+2)^2}$

Solution:

Given $f(t) = e^{-t}(t+2)^2$

$$= e^{-t}(t^2 + 4t + 4)$$

$$F(s) = L[f(t)]$$

$$= L[e^{-t}(t^2 + 4t + 4)]$$

$$= L[t^2 + 4t + 4]_{s\to s+1}$$

$$= [L(t^2) + 4L(t) + 4L(1)]_{s\to s+1}$$

$$= \left[\frac{2!}{s^3} + 4\frac{1}{s^2} + 4\frac{1}{s}\right]_{s\to s+1}$$

$$= \frac{2}{(s+1)^3} + 4\frac{1}{(s+1)^2} + 4\frac{1}{s+1}$$

$$sF(s) = \frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1}$$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$\lim_{t\to 0} f(t) = \lim_{t\to 0}[e^{-t}(t^2 + 4t + 4)]$$

$$= 4 \cdots (1)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left[\frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1}\right]$$

$$= \lim_{s\to\infty}\left[\frac{2s}{s^3\left(1+\frac{1}{s}\right)^3} + \frac{4s}{s^2\left(1+\frac{1}{s}\right)^2} + \frac{4s}{s\left(1+\frac{1}{s}\right)}\right]$$

$$= \lim_{s\to\infty}\left[\frac{2}{s^2\left(1+\frac{1}{s}\right)^3} + \frac{4}{s\left(1+\frac{1}{s}\right)^2} + \frac{4}{\left(1+\frac{1}{s}\right)}\right]$$

$$= 0 + 0 + 4$$

$$= 4 \cdots (2)$$

From (1) and (2), $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

∴ Initial value theorem is verified

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty}[e^{-t}(t^2 + 4t + 4)]$$

$$= 0 \cdots (3)$$

$$\lim_{s\to 0} sF(s) = \lim_{s\to 0}\left[\frac{2s}{(s+1)^3} + \frac{4s}{(s+1)^2} + \frac{4s}{s+1}\right]$$

$$= 0 \cdots (4)$$

From (3) and (4), $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

∴ Final value theorem is verified.

Example: If $L[f(t)] = \frac{1}{s(s+1)}$, find the $\lim_{t\to 0} f(t)$ and $\lim_{t\to\infty} f(t)$ using initial and final value theorems.

Solution:

Given $L[f(t)] = \frac{1}{s(s+1)} \cdots (1)$

$ie., F(s) = \frac{1}{s(s+1)} \Rightarrow sF(s) = \frac{1}{(s+1)}$

Initial value theorem is $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

$$= \lim_{s\to\infty} \frac{1}{(s+1)} = 0$$

Final value theorem is $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$= \lim_{s\to 0} \frac{1}{(s+1)} = 1$$

3.3(b) CONVOLUTION THEOREM

Definition: Convolution of two functions

The convolution of two functions $f(t)$ and $g(t)$ is denoted by $f(t) * g(t)$ and defined by

$f(t) * g(t) = \int_0^t f(u)g(t-u)du.$

State and prove Convolution theorem

Statement: If $L[f(t)] = F(s)$ and $L[g(t)] = G(s)$, then $L[f(t)] * L[g(t)] = F(s)G(s)$

Proof:

We have $f(t) * g(t) = \int_0^t f(u)g(t-u)du$

$$L[f(t)*g(t)] = \int_0^\infty [f(t)*g(t)]\, e^{-st}dt$$

$$= \int_0^\infty \int_0^t f(u)g(t-u)du e^{-st}dt$$

$$= \int_0^\infty \int_0^t f(u)g(t-u)e^{-st}dudt \cdots (1)$$

Now we have no change the order of integration.

$u = 0, u = t; t = 0, t = \infty$

Change of order is . Draw horizontal strip PQ

At P, $t = u$, At A $u = \infty$

$$L[f(t)*g(t)] = \int_0^\infty \int_u^\infty f(u)g(t-u)e^{-st}dtdu$$

$$= \int_0^\infty f(u)\left[\int_u^\infty g(t-u)e^{-st}dt\right]du \cdots (2)$$

Put $t - u = x \cdots (3)$

$t = u + x \Rightarrow dt = dx$

When $t = u; (3) \Rightarrow x = 0$

When $t = \infty; (3) \Rightarrow x = \infty$

$$(2) \Rightarrow L[f(t)*g(t)] = \int_0^\infty f(u)\left[\int_0^\infty g(x)e^{-s(u+x)}dx\right]du$$

$$= \int_0^\infty f(u)\left[\int_0^\infty g(x)e^{-su}e^{-sx}dx\right]du$$

$$= \int_0^\infty f(u)e^{-su}du \int_0^\infty g(x)e^{-sx}dx$$

$$= L[f(u)]L[g(x)]$$

$\therefore L[f(t)*g(t)] = F(s)G(s)$

Note: Convolution theorem is very useful to compute inverse Laplace transform of product of two terms

Convolution theorem is $L[f(t)*g(t)] = F(s)G(s)$

$$L^{-1}[F(s)G(s)] = f(t)*g(t)$$

$$L^{-1}[F(s)G(s)] = L^{-1}[F(s)]*L^{-1}[G(s)]$$

Problems under Convolution theorem

Example: Find $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$ using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\frac{1}{(s+a)}\right]*L^{-1}\left[\frac{1}{(s+b)}\right]$$

$$= e^{-at}*e^{-bt}$$

$$= \int_0^t e^{-au}e^{-b(t-u)}\,du$$

$$= e^{-bt}\int_0^t e^{-au}\,e^{bu}du$$

$$= e^{-bt}\int_0^t e^{(b-a)u}\,du$$

$$= e^{-bt}\left[\frac{e^{(b-a)u}}{b-a}\right]_0^t$$

$$= \frac{e^{-bt}}{b-a}\left[e^{(b-a)t} - 1\right]$$

$$= \frac{e^{-bt}}{b-a}\left[e^{bt-at} - 1\right]$$

$$= \frac{1}{b-a}\left[e^{-bt+bt-at} - e^{-bt}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = \frac{1}{b-a}\left[e^{-at} - e^{-bt}\right]$$

Example: Find the inverse Laplace transform $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{s}{(s^2+a^2)}\,\frac{s}{(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right] * L^{-1}\left[\frac{s}{(s^2+b^2)}\right]$$

$$= cosat * cosbt$$

$$= \int_0^t cosau\, cosb(t-u)du$$

$$= \int_0^t \frac{\cos(au+bt-bu)+\cos(au-bt+bu)}{2}\,du$$

$$= \frac{1}{2}\int_0^t(\cos(au+bt-bu)+\cos(au-bt+bu))\,du$$

$$= \frac{1}{2}\int_0^t[\cos(a-b)\,u+bt+\cos(a+b)\,u-bt]du$$

$$= \frac{1}{2}\left[\frac{sin[(a-b)u+bt]}{a-b}+\frac{sin[(a+b)u+bt]}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(at-bt+bt)}{a-b}+\frac{\sin(at-bt+bt)}{a+b}-\frac{sinbt}{a-b}+\frac{sinbt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b}+\frac{\sin at}{a+b}-\frac{sinbt}{a-b}+\frac{sinbt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{(a+b)sinat+(a-b)sinat-(a+b)sinbt+(a-b)sinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2asinat-2bsinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2(asinat-bsinbt)}{a^2-b^2}\right]$$

$$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{asinat-bsinbt}{a^2-b^2}$$

Example: Find the inverse Laplace transform $\frac{1}{(s^2+a^2)(s^2+b^2)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\frac{1}{(s^2+a^2)}\ \frac{1}{(s^2+b^2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+a^2)}\right] * L^{-1}\left[\frac{1}{(s^2+b^2)}\right]$$

$$= \frac{1}{a} sinat * \frac{1}{b} sinbt$$

$$= \frac{1}{ab}\int_0^t sinau\ sinb(t-u)du$$

$$= \frac{1}{ab}\int_0^t \frac{\cos(au-bt+bu)-\cos(au+bt-bu)}{2} du$$

$$= \frac{1}{2ab}\int_0^t (\cos(au-bt+bu)-\cos(au+bt-bu))\, du$$

$$= \frac{1}{2}\int_0^t [\cos[(\mathrm{a}+\mathrm{b})\mathrm{u}-\mathrm{bt}]-\cos[(\mathrm{a}-\mathrm{b})\mathrm{u}+\mathrm{bt}]]du$$

$$= \frac{1}{2ab}\left[\frac{sin[(a+b)u-bt]}{a+b} - \frac{sin[(a-b)u+bt]}{a-b}\right]_0^t$$

$$= \frac{1}{2ab}\left[\frac{\sin(at+bt-bt)}{a+b} - \frac{\sin(at-bt+bt)}{a-b} + \frac{sinbt}{a+b} + \frac{sinbt}{a-b}\right]$$

$$= \frac{1}{2ab}\left[\frac{\sin at}{a+b} - \frac{\sin at}{a-b} - \frac{sinbt}{a+b} + \frac{sinbt}{a-b}\right]$$

$$= \frac{1}{2ab}\left[\frac{(a-b)sinat-(a+b)sinat+(a-b)sinbt+(a+b)sinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2ab}\left[\frac{-2bsinat+2asinbt}{a^2-b^2}\right]$$

$$= \frac{1}{2ab}\left[\frac{2(asinbt-bsinat)}{a^2-b^2}\right]$$

$$\therefore L^{-1}\left[\frac{1}{(s^2+a^2)(s^2+b^2)}\right] = \frac{asinbt-bsinat}{ab(a^2-b^2)}$$

Example: Find the inverse Laplace transform $\frac{s}{(s^2+4)(s^2+9)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] = L^{-1}\left[\frac{1}{(s^2+4)}\ \frac{s}{(s^2+9)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+4)}\right] * L^{-1}\left[\frac{s}{(s^2+9)}\right]$$

$$= \frac{1}{2} sin2t * cos3t$$

$$= \frac{1}{2}\int_0^t sin2u\ cos3(t-u)du$$

$$= \frac{1}{2}\int_0^t \frac{\sin(2u+3t-3u)+\sin(2u-3t+3u)}{2} du$$

$$= \frac{1}{4}\int_0^t [\sin(3\mathrm{t}-u)+\sin(5\mathrm{u}-3\mathrm{t})]du$$

$$= \frac{1}{4}\left[\frac{-\cos(3t-u)}{-1} - \frac{\cos(5u-3t)}{5}\right]_0^t$$

$$= \frac{1}{4}\left[\frac{\cos(3\mathrm{t}-\mathrm{t})}{1} - \frac{\cos(5\mathrm{t}-3t)}{5} - \frac{cos3t}{1} + \frac{cos3t}{5}\right]$$

$$= \frac{1}{4}\left[cos2t - \frac{\cos 2t}{5} - cos3t + \frac{cos3t}{5}\right]$$

$$= \frac{1}{4}\left[\frac{5cos2t - cos2t - 5cos3t + cos3t}{5}\right]$$

$$= \frac{1}{20}[4cos2t - 4cos3t]$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+4)(s^2+9)}\right] = \frac{cos2t - cos3t}{5}$$

Example: Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{(s^2+a^2)}\ \frac{s}{(s^2+a^2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+a^2)}\right] * L^{-1}\left[\frac{s}{(s^2+a^2)}\right]$$

$$= \frac{1}{a} sinat * cosat$$

$$= \frac{1}{a}\int_0^t sinau\ cosa(t-u)du$$

$$= \frac{1}{a}\int_0^t \frac{\sin(au+at-au)+\sin(au-at+au)}{2}du$$

$$= \frac{1}{2a}\int_0^t [sinat + \sin(2au - at)]du$$

$$= \frac{1}{2a}\left[\int_0^t sinatdu + \int_0^t \sin(2au-at)\,du\right]$$

$$= \frac{1}{2a}\left[sinat\int_0^t du + \int_0^t \sin(2au-at)\,du\right]$$

$$= \frac{1}{2a}\left[sinat(u)_0^t - \left(\frac{\cos(2au-at)}{2a}\right)_0^t\right]$$

$$= \frac{1}{2a}\left[tsinat - \frac{\cos(2at-at)}{2a} + \frac{cosat}{2a}\right]$$

$$= \frac{1}{2a}\left[tsinat - \frac{\cos at}{2a} + \frac{cosat}{2a}\right]$$

$$= \frac{1}{2a}\ tsinat$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{tsinat}{2a}$$

Example:Find $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{1}{(s^2+a^2)}\ \frac{1}{(s^2+a^2)}\right]$$

$$= L^{-1}\left[\frac{1}{(s^2+a^2)}\right] * L^{-1}\left[\frac{1}{(s^2+a^2)}\right]$$

$$= \frac{1}{a} sinat * \frac{1}{a} sinat$$

$$= \frac{1}{a^2}\int_0^t sinau\ sina(t-u)du$$
$$= \frac{1}{a^2}\int_0^t \frac{\cos(au-at+au)-\cos(au+at-au)}{2}du$$
$$= \frac{1}{2a^2}\int_0^t [\cos(2au-at)-cosat]du$$
$$= \frac{1}{2a^2}\left[\int_0^t \cos(2au-at)du-\int_0^t cosat\ du\right]$$
$$= \frac{1}{2a^2}\left[\int_0^t \cos(2au-at)du-cosat\int_0^t du\right]$$
$$= \frac{1}{2a^2}\left[\left(\frac{\sin(2au-at)}{2a}\right)_0^t-cosat(u)_0^t\right]$$
$$= \frac{1}{2a^2}\left[\frac{\sin(2at-at)}{2a}-\frac{\sin(-at)}{2a}-tcosat\right]$$
$$= \frac{1}{2a^2}\left[\frac{\sin at}{2a}+\frac{sinat}{2a}-tcosat\right]$$
$$= \frac{1}{2a^2}\left[\frac{2\sin at}{2a}-tcosat\right]$$
$$\therefore L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]=\frac{1}{2a^2}\left[\frac{\sin at}{a}-tcosat\right]$$

Example: Find $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right]=L^{-1}\left[\frac{s}{(s^2+a^2)}\ \frac{s}{(s^2+a^2)}\right]$$
$$= L^{-1}\left[\frac{s}{(s^2+a^2)}\right]*L^{-1}\left[\frac{s}{(s^2+a^2)}\right]$$
$$= cosat * cosat$$
$$= \int_0^t cosau\ cosa(t-u)du$$
$$= \int_0^t \frac{\cos(au+at-au)+\cos(au-at+au)}{2}du$$
$$= \frac{1}{2}\int_0^t [cosat+\cos(2au-at)]du$$
$$= \frac{1}{2}\left[\int_0^t cosatdu+\int_0^t \cos(2au-at)\,du\right]$$
$$= \frac{1}{2}\left[cosat\int_0^t du+\int_0^t \cos(2au-at)\,du\right]$$
$$= \frac{1}{2}\left[cosat(u)_0^t+\left(\frac{\sin(2au-at)}{2a}\right)_0^t\right]$$
$$= \frac{1}{2}\left[tcosat+\frac{\sin(2at-at)}{2a}+\frac{sinat}{2a}\right]$$
$$= \frac{1}{2}\left[tcosat+\frac{\sin at}{2a}+\frac{sinat}{2a}\right]$$
$$= \frac{1}{2}\left[tcosat+\frac{2\sin at}{2a}\right]$$

$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2}\left[tcosat + \frac{\sin at}{a}\right]$

Example: Find $L^{-1}\left[\frac{s^2}{(s^2+4)^2}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s^2}{(s^2+2^2)^2}\right] = L^{-1}\left[\frac{s}{(s^2+2^2)}\,\frac{s}{(s^2+2^2)}\right]$$
$$= L^{-1}\left[\frac{s}{(s^2+2^2)}\right] * L^{-1}\left[\frac{s}{(s^2+2^2)}\right]$$
$$= cos2t * cos2t$$
$$= \int_0^t cos2u\, cos2(t-u)du$$
$$= \int_0^t \frac{\cos(2u+2t-2u)+\cos(2u-2t+2u)}{2}du$$
$$= \frac{1}{2}\int_0^t [cos2t + \cos(4u-2t)]du$$
$$= \frac{1}{2}\left[\int_0^t cos2tdu + \int_0^t \cos(4u-2t)\,du\right]$$
$$= \frac{1}{2}\left[cos2t\int_0^t du + \int_0^t \cos(4u-2t)\,du\right]$$
$$= \frac{1}{2}\left[cos2t(u)_0^t + \left(\frac{\sin(4u-2t)}{4}\right)_0^t\right]$$
$$= \frac{1}{2}\left[tcos2t + \frac{\sin(4t-2t)}{4} - \frac{\sin(-2t)}{4}\right]$$
$$= \frac{1}{2}\left[tcos2t + \frac{\sin 2t}{4} + \frac{sin2t}{4}\right]$$
$$= \frac{1}{2}\left[tcos2t + \frac{2\sin 2t}{4}\right]$$

$\therefore L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2}\left[tcos2t + \frac{\sin 2t}{2}\right]$

Example: Find $L^{-1}\left[\frac{1}{s(s^2+4)}\right]$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{s(s^2+4)}\right] = L^{-1}\left[\frac{1}{s}\,\frac{1}{s^2+4}\right]$$
$$= L^{-1}\left[\frac{1}{s}\right] * L^{-1}\left[\frac{1}{s^2+4}\right]$$
$$= 1 * \frac{\sin 2t}{2}$$
$$= \frac{\sin 2t}{2} * 1$$
$$= \int_0^t \frac{\sin 2u\,(1)}{2}du$$
$$= \left[\frac{-cos2u}{4}\right]_0^t = \frac{1}{4}(-cos2t + 1)$$

$$= \frac{1}{4}(1 - cos2t)$$

Example: Find the inverse Laplace transform $\frac{s+2}{(s^2+4s+13)^2}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right] = L^{-1}\left[\frac{s+2}{s^2+4s+13}\ \frac{1}{s^2+4s+13}\right]$$

$$= L^{-1}\left[\frac{s+2}{s^2+4s+13}\right] * L^{-1}\left[\frac{1}{s^2+4s+13}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] * L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$= e^{-2t}L^{-1}\left[\frac{s}{s^2+9}\right] * e^{-2t}L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= e^{-2t}cos3t * \frac{e^{-2t}sin3t}{3}$$

$$= \int_0^t e^{-2u}cos3u\ e^{-2(t-u)}\ \frac{sin3(t-u)}{3}\,du$$

$$= \int_0^t e^{-2u}cos3u\ e^{-2t+2u}\ \frac{sin(3t-3u)}{3}\,du$$

$$= \frac{1}{3}\int_0^t e^{-2u-2t+2u}cos3u\,\sin(3t-3u)\,du$$

$$= \frac{e^{-2t}}{3}\int_0^t \frac{\sin(3u+3t-3u)-\sin(3u-3t+3u)}{2}\,du$$

$$= \frac{e^{-2t}}{6}\int_0^t [sin3t - \sin(6u-3t)]du$$

$$= \frac{e^{-2t}}{6}\left[\int_0^t sin3tdu - \int_0^t \sin(6u-3t)\,du\right]$$

$$= \frac{e^{-2t}}{6}\left[sin3t\int_0^t du - \int_0^t \sin(6u-3t)\,du\right]$$

$$= \frac{e^{-2t}}{6}\left[sin3t(u)_0^t + \left(\frac{\cos(6u-3t)}{6}\right)_0^t\right]$$

$$= \frac{e^{-2t}}{6}\left[tsin3t + \frac{\cos(6t-3t)}{6} - \frac{\cos(-3t)}{6}\right]$$

$$= \frac{e^{-2t}}{6}\left[tsin3t + \frac{\cos3t}{6} - \frac{cos3t}{6}\right]$$

$$= \frac{e^{-2t}}{6}\ tsin3t$$

$$\therefore L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right] = \frac{e^{-2t}}{6}\ tsin3t$$

Example: Find the inverse Laplace transform $\frac{1}{(s+1)(s^2+4)}$ by using convolution theorem.

Solution:

$$L^{-1}\left[\frac{1}{(s^2+4)(s+1)}\right] = L^{-1}\left[\frac{1}{s+1}\ \frac{1}{s^2+4}\right]$$

$$= L^{-1}\left[\frac{1}{s+1}\right] * L^{-1}\left[\frac{1}{s^2+4}\right]$$

$= e^{-t} * cos2t$

$= \int_0^t e^{-(t-u)} cos2u\, du$

$= e^{-t} \int_0^t e^{u} cos2u\, du$

$$\because \int e^{at} cosbtdt = \frac{e^{at}}{a^2+b^2}(acosbt + bsinbt)$$

$= e^{-t} \left[\frac{e^u}{1^2+2^2}(cos2u + 2sin2u)\right]_0^t$

$= \frac{e^{-t}}{5}[e^t(cos2t + 2sin2t) - e^0(cos0 - 0)]$

$= \frac{e^{-t}}{5}[e^t(cos2t + 2sin2t) - 1]$

$\therefore L^{-1}\left[\frac{1}{(s^2+4)(s+1)}\right] = \frac{e^{-t}}{5}[e^t(cos2t + 2sin2t) - 1]$