5.4 Change of variables in Double and Triple integrals:

Evaluation of double integrals by changing Cartesian to polar co- ordinates:

Working rule:

Step:1

Check the given order whether it is correct or not.

Step:2

Write the equations by using given limits.

Step:3

By using the equations sketch the region of integration.

Step:4

Replacement: put $x = r\cos\theta$, $y = r\sin\theta$, $x^2 + y^2 = r^2$ and $dxdy = rdrd\theta$

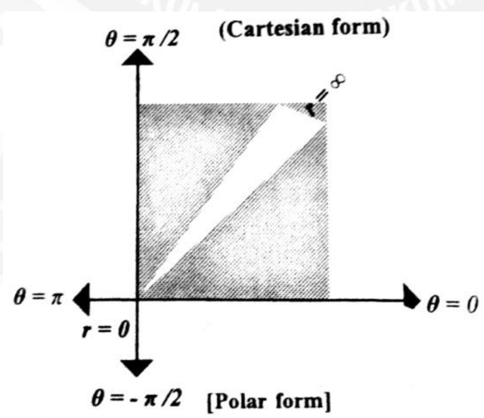
Step:5

Find r limits(draw radial strip inside the region) and θ limits and evaluate the integral.

Example:

Change into polar co-ordinates and then evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dydx$

Solution:



Given order $dydx$ is in correct form.

Given limits are $y:\ 0 \to \infty$, $x:\ 0 \to \infty$

Equations are $y = 0$, $y = \infty$, $x = 0$, $x = \infty$

Replacement:

Put $x^2 + y^2 = r^2$, dydx = $rdrd\theta$

Limits:

$r : 0 \to \infty$, $\theta: 0 \to \frac{\pi}{2}$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dydx = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} rdrd\theta$$

Substitution: Put $r^2 = t$, if r = 0 ⇒ $t = 0$, $r = \infty \Rightarrow t = \infty$

$2rdr = dt$ $t: 0 \to \infty$

$rdr = \frac{dt}{2}$

$$\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} rdrd\theta = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-t} \frac{dt}{2} d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \left[\frac{e^{-t}}{-1}\right]_0^\infty d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (-e^{-\infty} + e^0) d\theta$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} (0 + 1) d\theta \qquad (\because e^{-\infty} = 0, e^0 = 1)$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{1}{2} (\theta)_0^{\frac{\pi}{2}}$$

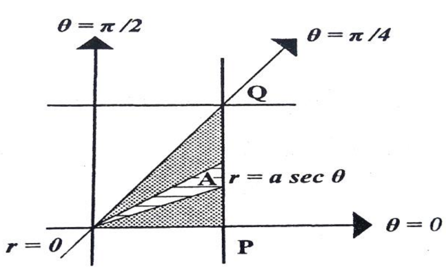
$$= \frac{1}{2} \left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{4}$$

Example:

Change into polar co-ordinates and then evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2} dydx$

Solution:



Given order $dxdy$ is in correct form.

Given limits are $x : y \to a$, $y : 0 \to a$

Equations are $x = y$, $x = a$, $y = 0$, $y = a$

Replacement:

Put $x = r\cos\theta$, $x^2 + y^2 = r^2$, $dxdy = rdrd\theta$

Limits: $r : 0 \to \frac{a}{\cos\theta}$, $\theta : 0 \to \frac{\pi}{4}$

$$\int_0^a \int_y^a \frac{x}{x^2+y^2} dydx = \int_0^{\frac{\pi}{4}} \int_0^{\frac{a}{\cos\theta}} \frac{r\cos\theta}{r^2} rdrd\theta$$

$$= \int_0^{\frac{\pi}{4}} [r\cos\theta]_0^{\frac{a}{\cos\theta}} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{a}{\cos\theta}\cos\theta - 0\right) d\theta$$

$$= a\int_0^{\frac{\pi}{4}} d\theta$$

$$= a(\theta)_0^{\frac{\pi}{4}}$$

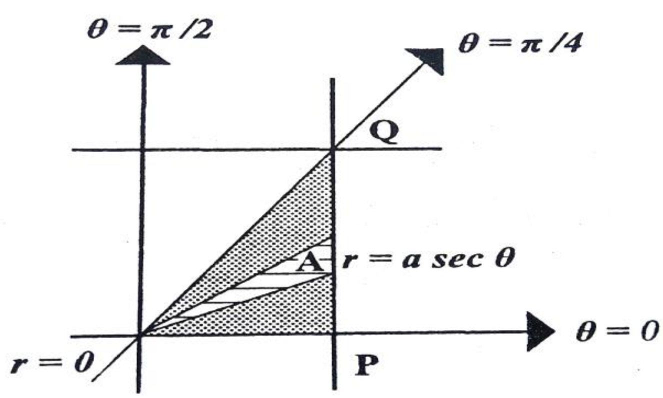
$$= a\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{a\pi}{4}$$

Example:

Evaluate $\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}}$ dxdy by changing into polar co-ordinates.

Solution:



Given order $dxdy$ is in correct form.

Given limits are $x : y \to a$, $y : 0 \to a$

Equations are $x = y$, $x = a$, $y = 0$, $y = a$

Replacement:

Put $x^2 = r^2 cos^2\theta$, $x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2}$, $dxdy = rdrd\theta$

Limits: $r : 0 \to \frac{a}{cos\theta}$, $\theta : 0 \to \frac{\pi}{4}$

$$\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2+y^2}} dxdy = \int_0^{\frac{\pi}{4}} \int_0^{\frac{a}{cos\theta}} \frac{r^2 cos^2\theta}{r} rdrd\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[\frac{r^3}{3} cos^2\theta\right]_0^{\frac{a}{cos\theta}} d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{a^3}{3cos^3\theta} cos^2\theta - 0\right) d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{4}} \frac{1}{cos^3\theta} cos^2\theta \, d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{4}} \frac{1}{cos\theta} d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{4}} sec\theta \, d\theta$$

$$= \frac{a^3}{3} (log(sec\theta + tan\theta))_0^{\frac{\pi}{4}}$$

$$= \frac{a^3}{3} \left[log\left(sec\frac{\pi}{4} + tan\frac{\pi}{4}\right) - log(sec0 + tan0)\right]$$

$$= \frac{a^3}{3} \left[log\left(\sqrt{2} + 1\right) - log(1 - 0)\right]$$

$$= \frac{a^3}{3} log\left(\sqrt{2} + 1\right)$$

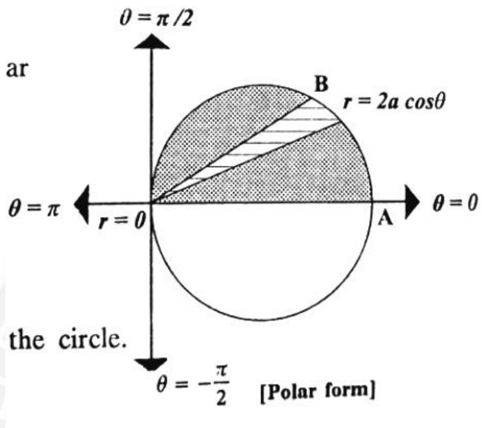
Note:

1. $x^2 + y^2 = r^2cos^2\theta + r^2sin^2\theta = r^2$
2. $\int_0^{\frac{\pi}{2}} cos^2\theta d\theta = \int_0^{\frac{\pi}{2}} sin^2\theta d\theta = \frac{1}{2} \times \frac{\pi}{2}$
3. $\int_0^{\frac{\pi}{2}} cos^4\theta d\theta = \int_0^{\frac{\pi}{2}} sin^4\theta d\theta = \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$
4. $\int_0^{\frac{\pi}{2}} cos^2\theta sin^2\theta d\theta = \frac{1}{4} \times \frac{1}{2} \times \frac{\pi}{2}$

Example:

By changing into polar co-ordinates and evaluate $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)\,dydx$

Solution:



Given order $dydx$ is in correct form.

Given limits are $y : 0 \to \sqrt{2ax - x^2}$, $x : 0 \to 2a$

Equations are $y = 0$, $y = \sqrt{2ax - x^2}$, $x = 0$, $x = 2a$

$$y^2 = 2ax - x^2$$

$x^2 + y^2 - 2ax = 0$ is a circle with centre (a,0) and radius 'a'.

Replacement:

Put $x^2 + y^2 = r^2$, $dxdy = rdrd\theta$

Limits: $r : 0 \to 2acos\theta$, $\theta : 0 \to \frac{\pi}{2}$

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)\,dydx = \int_0^{\frac{\pi}{2}}\int_0^{2acos\theta} r^2 \times rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{2acos\theta} r^3 drd\theta$$

$$= \int_0^{\frac{\pi}{2}}\left[\frac{r^4}{4}\right]_0^{2acos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left(\frac{2^4a^4cos^4\theta}{4} - 0\right) d\theta$$

$$= 4a^4 \int_0^{\frac{\pi}{2}} cos^4\theta \, d\theta$$

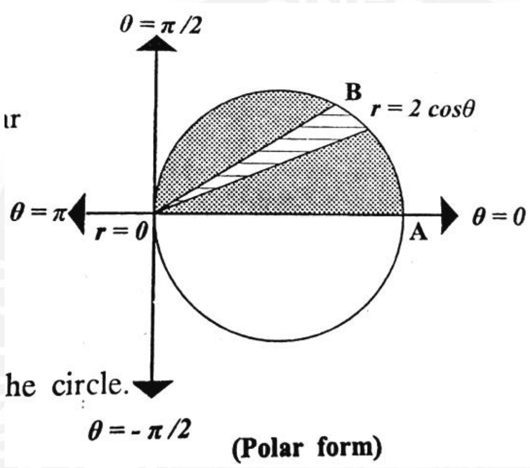
$= 4a^4 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$ $\quad (\because \int_0^{\frac{\pi}{2}} \cos^4\theta d\theta = \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2})$

$= \frac{3\pi a^4}{4}$

Example:

By changing into polar co-ordinates and evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2} dxdy$

Solution:



Given order $dxdy$ is in incorrect form.

The correct form is $dydx \Rightarrow \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2} dydx$

Given limits are $y : 0 \to \sqrt{2x - x^2}$, $x : 0 \to 2$

Equations are $y = 0$, $y = \sqrt{2x - x^2}$, $x = 0$, $x = 2$

$y^2 = 2x - x^2$

$x^2 + y^2 - 2x = 0$ is a circle with centre (1,0) and radius ‘1’.

Replacement:

Put $x = r\cos\theta$, $x^2 + y^2 = r^2$, $dxdy = rdrd\theta$

Limits: $r : 0 \to 2\cos\theta$, $\theta: 0 \to \frac{\pi}{2}$

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2} dydx = \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{r\cos\theta}{r^2} \times rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} [r\cos\theta]_0^{2\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} (2\cos^2\theta - 0)\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} cos^2\theta \, d\theta$$

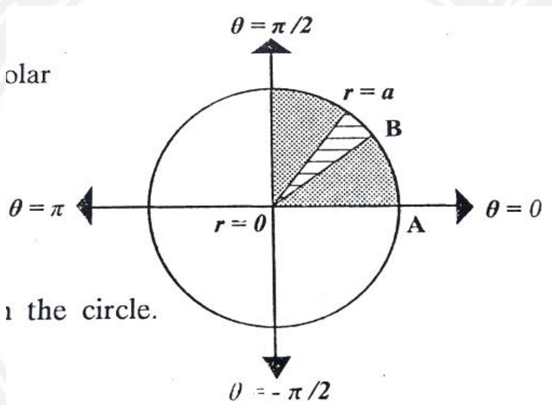
$$= 2 \times \frac{1}{2} \times \frac{\pi}{2} \qquad (\because \int_0^{\frac{\pi}{2}} cos^2\theta d\theta = \frac{1}{2} \times \frac{\pi}{2})$$

$$= \frac{\pi}{2}$$

Example:

By changing into polar co-ordinates and evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dydx$

Solution:



Given order $dxdy$ is in correct form.

Given limits are $y : 0 \rightarrow \sqrt{a^2-x^2}$, $x : 0 \rightarrow a$

Equations are $y = 0$, $y = \sqrt{a^2-x^2}$, $x = 0$, $x = a$

$$y^2 = a^2 - x^2$$

$x^2 + y^2 = a^2$ is a circle with centre (0,0) and radius ‘a’.

Replacement:

Put $x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2+y^2}$, $dydx = rdrd\theta$

Limits: $r : 0 \rightarrow a$, $\theta : 0 \rightarrow \frac{\pi}{2}$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dydx = \int_0^{\frac{\pi}{2}} \int_0^a r \times r dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^a r^2 dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{r^3}{3}\right]_0^a d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{a^3}{3} - 0\right) d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{a^3}{3} (\theta)_0^{\frac{\pi}{2}}$$

$$= \frac{a^3}{3} \left(\frac{\pi}{2} - 0\right)$$

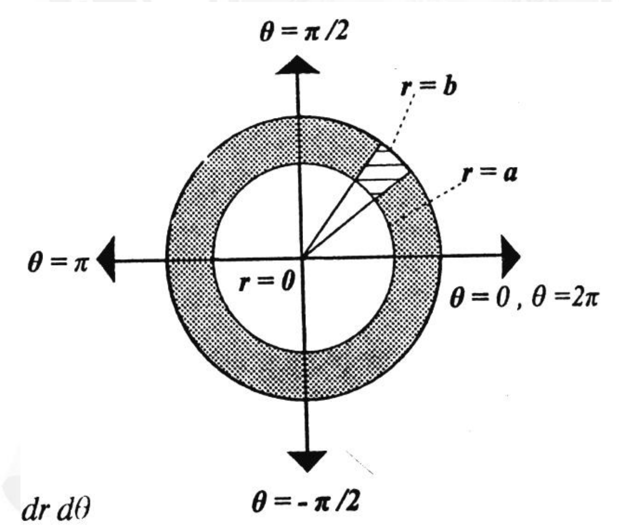
$$= \frac{\pi a^3}{6}$$

Example:

Evaluate $\iint \frac{x^2y^2}{x^2+y^2} dxdy$ over the annular region between the circles $x^2 + y^2 = a^2$ and

$x^2 + y^2 = b^2$ (b>a) by transforming into polar co-ordinates.

Solution:



Replacement:

Put $x^2 = r^2\cos^2\theta, y^2 = r^2\sin^2\theta$

$x^2 + y^2 = r^2, dxdy = rdrd\theta$

Given the region is between the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$

Limits: $r : a \to b$, $\theta: 0 \to 2\pi$

$$\therefore \iint \frac{x^2y^2}{x^2+y^2} dxdy = \int_0^{2\pi} \int_a^b \frac{r^2cos^2\theta \times r^2sin^2\theta}{r^2} \times rdrd\theta$$

$$= \int_0^{2\pi}\int_a^b \frac{r^5\cos^2\theta\times\sin^2\theta}{r^2}\times dr d\theta$$

$$= \int_0^{2\pi}\int_a^b r^3\cos^2\theta\times\sin^2\theta\, dr d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^4}{4}\right]_a^b \cos^2\theta\times\sin^2\theta\, d\theta$$

$$= \frac{1}{4}\int_0^{2\pi}(b^4-a^4)\cos^2\theta\times\sin^2\theta\, d\theta$$

$$= \frac{(b^4-a^4)}{4}\int_0^{2\pi}\cos^2\theta\times\sin^2\theta\, d\theta$$

$$= \frac{(b^4-a^4)}{4}4\times\int_0^{\frac{\pi}{2}}\cos^2\theta\times\sin^2\theta\, d\theta \quad (\because \int_0^{2\pi} = 4\int_0^{\frac{\pi}{2}})$$

$$= (b^4-a^4)\times\int_0^{\frac{\pi}{2}}\cos^2\theta\times\sin^2\theta\, d\theta$$

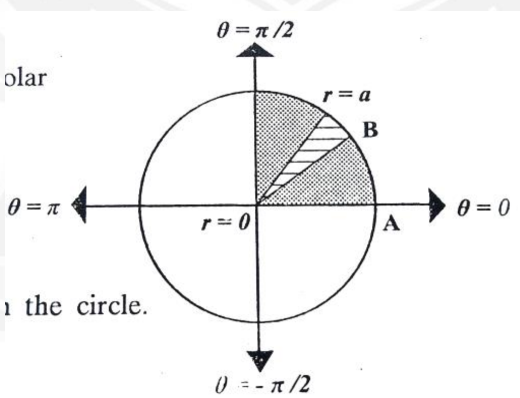
$$= (b^4-a^4)\times\frac{1}{4}\times\frac{1}{2}\times\frac{\pi}{2} \quad (\because \int_0^{\frac{\pi}{2}}\cos^2\theta\sin^2\theta d\theta = \frac{1}{4}\times\frac{1}{2}\times\frac{\pi}{2})$$

$$= \frac{\pi(b^4-a^4)}{16}$$

Example:

Evaluate $\int_0^a\int_0^{\sqrt{a^2-x^2}}\sqrt{a^2-x^2-y^2}\,dydx$ by transforming into polar co-ordinates.

Solution:



Given order $dydx$ is in correct form.

Given limits are $y : 0\to \sqrt{a^2-x^2}$, $x : 0\to a$

Equations are $y = 0$, $y = \sqrt{a^2-x^2}$, $x = 0$, $x = a$

$$y^2 = a^2 - x^2$$

$x^2 + y^2 = a^2$ is a circle with centre (0,0) and radius ‘a’.

Replacement:

Put $a^2 - x^2 - y^2 = a^2 - (x^2 + y^2) = a^2 - r^2$, $dydx = rdrd\theta$

$\therefore \sqrt{a^2 - x^2 - y^2} = \sqrt{a^2 - r^2}$

Limits: $r : 0 \to a$, $\theta : 0 \to \frac{\pi}{2}$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dydx = \int_0^{\frac{\pi}{2}} \int_0^a \sqrt{a^2 - r^2}\, rdrd\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(\int_0^a \sqrt{a^2 - r^2}\, rdr\right) d\theta$$

Substitution:

Put $a^2 - r^2 = t$ if $r = 0 \Rightarrow t = a^2$

$-2rdr = dt$ if $r = a \Rightarrow t = 0$

$rdr = -\frac{dt}{2}$

$\therefore t : a^2 \to 0$

$$\int_0^{\frac{\pi}{2}} \left(\int_0^a \sqrt{a^2 - r^2}\, rdr\right) d\theta = \int_0^{\frac{\pi}{2}} \left[\int_{a^2}^0 \sqrt{t}\left(\frac{-dt}{2}\right)\right] d\theta$$

$$= \frac{-1}{2} \int_0^{\frac{\pi}{2}} \left[\int_{a^2}^0 \sqrt{t}\, dt\right] d\theta$$

$$= \frac{-1}{2} \int_0^{\frac{\pi}{2}} \left[\int_{a^2}^0 t^{\frac{1}{2}}\, dt\right] d\theta$$

$$= \frac{-1}{2} \int_0^{\frac{\pi}{2}} \left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_{a^2}^0 d\theta$$

$$= -\frac{1}{2} \times \frac{2}{3} \int_0^{\frac{\pi}{2}} \left[t^{\frac{3}{2}}\right]_{a^2}^0 d\theta$$

$$= -\frac{1}{3} \int_0^{\frac{\pi}{2}} \left(0 - (a^2)^{\frac{3}{2}}\right) d\theta$$

$$= -\frac{1}{3} \int_0^{\frac{\pi}{2}} -(a^3)\, d\theta$$

$$= \frac{a^3}{3} \int_0^{\frac{\pi}{2}} d\theta$$

$$= \frac{a^3}{3} (\theta)_0^{\frac{\pi}{2}}$$

$$= \frac{a^3}{3}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi a^3}{6}$$

Exercise:

Evaluate the following by changing into polar co-ordinates.

1. $\int_0^{2a}\int_0^{\sqrt{2x-x^2}} dydx$ **Ans:** $\frac{\pi a^2}{2}$

2. $\int_0^{a}\int_0^{\sqrt{a^2-x^2}} (x^2+y^2)dydx$ **Ans:** $\frac{\pi a^4}{8}$

3. $\int_0^{1}\int_{x^2}^{2-x} xy\, dxdy$ **Ans:** $\frac{3}{8}$

4. $\int_0^{a}\int_y^{a} \frac{x}{x^2+y^2} dxdy$ **Ans:** $\frac{\pi a}{4}$

5. $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} \frac{x}{x^2+y^2} dxdy$ **Ans:** $\frac{\pi a}{2}$

6. $\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} (x^2+y^2)dydx$ **Ans:** $\frac{\pi a^4}{4}$

7. $\int_0^{a}\int_0^{\sqrt{a^2-x^2}} (x^2y+y^3)dxdy$ **Ans:** $\frac{a^5}{5}$

8. $\int_0^{1}\int_0^{\sqrt{2x-x^2}} (x^2+y^2)dydx$ **Ans:** $\frac{3\pi}{8} - 1$

9. $\iint \frac{x^2y^2}{x^2+y^2}$ dxdy over the annular region between the circles $x^2 + y^2 = 16$ and $x^2 + y^2 = 4$

Ans: 15π

10. $\iint \frac{xy}{x^2+y^2}$ dxdy over the positive quadrant of the circle $x^2 + y^2 = a^2$ **Ans:** $\frac{a^3}{6}$

Change of Variables in Triple Integral

Change of variables from Cartesian co- ordinates to cylindrical co – ordinates.

To convert from Cartesian to cylindrical polar coordinates system we have the following transformation.

$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, z)} = r$$

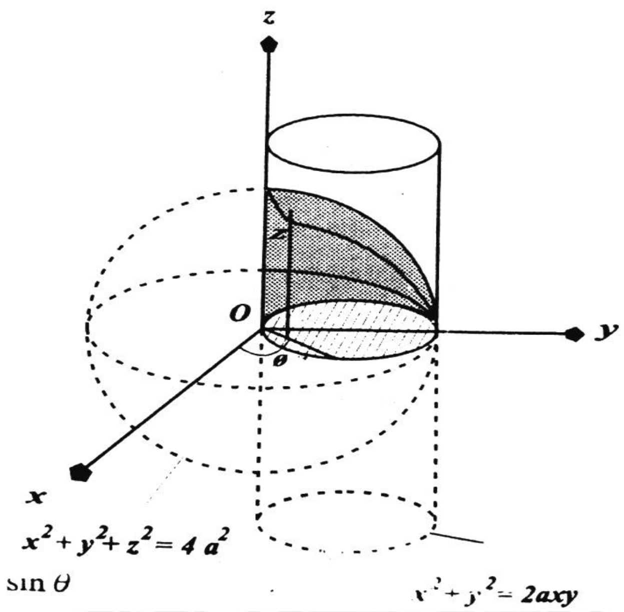
Hence the integral becomes

$$\iiint f(x, y, z)\, dzdydx = \iiint f(r, \theta, z)\, dzdrd\theta$$

Example:

Find the volume of a solid bounded by the spherical surface $x^2 + y^2 + z^2 = 4a^2$ and the cylinder $x^2 + y^2 - 2ay = 0$.

Solution:



Cylindrical co – ordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

The equation of the sphere $x^2 + y^2 + z^2 = 4a^2$

$$r^2\cos^2\theta + r^2\sin^2\theta + z^2 = 4a^2$$

$$r^2 + z^2 = 4a^2$$

And the cylinder $x^2 + y^2 - 2ay = 0$

$$x^2 + y^2 = 2ay$$

$$r^2\cos^2\theta + r^2\sin^2\theta = 2a\, r\sin\theta$$

$$r^2 = 2ar\sin\theta$$

$$r = 2a\sin\theta$$

Hence, the required volume,

$$\text{Volume} = \int\int\int dx\, dy\, dz$$

$$= \int\int\int r\, d\theta\, dr\, dz$$

$$= 4\int_0^{\pi/2}\int_0^{2a\sin\theta}\int_0^{\sqrt{4a^2-r^2}} r\, dz\, dr\, d\theta$$

$$= 4\int_0^{\pi/2}\int_0^{2a\sin\theta} r\sqrt{4a^2-r^2}\, dr\, d\theta$$

$$= 4\int_0^{\pi/2}\left[-\frac{1}{3}(4a^2-r^2)^{3/2}\right]_0^{2a\sin\theta} d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2}\left[-(4a^2-4a^2\sin^2\theta)^{3/2}+8a^3\right] d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2}(-8a^3\cos^3\theta+8a^3)\, d\theta$$

$$= \frac{4}{3}8a^3\int_0^{\pi/2}(1-\cos^3\theta)\, d\theta$$

$$= \frac{32\,a^3}{3}\left[\frac{\pi}{2}-\frac{2}{3}\right] \text{cubic units}$$

Example:

Find the volume of the portion of the cylinder $x^2+y^2=1$ intercepted between the plane $x=0$ and the paraboloid $x^2+y^2=4-z$.

Solution:

Cylindrical co – ordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

Given $x^2+y^2=1$

$$r^2\cos^2\theta + r^2\sin^2\theta = 1$$

$$r^2 = 1$$

$$r = \pm 1$$

Given $x^2+y^2=4-z$

$$r^2\cos^2\theta + r^2\sin^2\theta = 4-z$$

$$r^2 = 4-z$$

$$z = 4-r^2$$

Hence the required volume

$$\text{Volume} = \int\int\int r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \int_0^{4-r^2} r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r\,[z]_0^{4-r^2}\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r\,(4 - r^2)\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 (4r - r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{4r^2}{2} - \frac{r^4}{4}\right] d\theta$$

$$= \int_0^{2\pi} \left[\left(2 - \frac{1}{4}\right) - (0 - 0)\right] d\theta$$

$$= \int_0^{2\pi} \frac{7}{4}\, d\theta$$

$$= \frac{7}{4}\,[\theta]_0^{2\pi}$$

$$= \frac{7}{4}\,[2\pi - 0] = \frac{7}{2}\pi \text{ cubic.units}$$

Example:

Find the volume bounded by the paraboloid $x^2 + y^2 = az$, and the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$

Solution:

Cylindrical co – ordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

The equation of the sphere $x^2 + y^2 += az$

$$r^2\cos^2\theta + r^2\sin^2\theta = az$$

$$r^2 = az$$

And the cylinder $x^2 + y^2 = 2ay$

$$r^2\cos^2\theta + r^2\sin^2\theta = 2a\, r\sin\theta$$

$$r^2 = 2ar\sin\theta$$

$$r = 2a\sin\theta$$

Hence, the required volume,

$$\text{Volume} = \int\int\int dx\,dy\,dz$$

$$= \int\int\int r\,d\theta\,dr\,dz$$

$$= \int_0^{\pi} \int_0^{2a\sin\theta} \int_0^{\frac{r^2}{a}} r\,dz\,dr\,d\theta$$

$$= \int_0^{\pi} \int_0^{2a\sin\theta} [z]_0^{\frac{r^2}{a}} r dr d\theta$$

$$= \int_0^{\pi} \int_0^{2a\sin\theta} \left[\frac{r^3}{a}\right] dr d\theta$$

$$= \frac{1}{a}\int_0^{\pi} \left[\frac{r^4}{4}\right]_0^{2a\sin\theta} d\theta$$

$$= \frac{1}{a}\int_0^{\pi} \frac{16a^4\sin^4\theta}{4} d\theta$$

$$= 4a^3 \times 2\int_0^{\pi/2} \sin^4\theta d\theta$$

$$= 4a^3 \times 2\,\frac{3}{4}\frac{1}{2}\frac{\pi}{2} \quad = \frac{3\pi a^3}{2}$$

Change of variables from Cartesian Co – ordinates to spherical Polar Co – ordinates

To convert from Cartesian to spherical polar co-ordinates system we have the following transformation

$$x = rsin\theta cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

$$J = \frac{\partial(x, y, z)}{\partial(r, \theta, \varphi)} = r^2\sin\theta$$

Hence the integral becomes

$$\iiint f(x, y, z)\,dzdydx = \iiint f(r, \theta, z) r^2\sin\theta\,drd\theta d\varphi$$

Example:

Evaluate $\int\int\int \frac{1}{\sqrt{1 - x^2 - y^2 - z^2}}$ dx dy dz over the region bounded by the sphere $x^2 + y^2 + z^2 = 1$.

Solution:

Let us transform this integral in spherical polar co – ordinates by taking

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$dx\,dy\,dz = (r^2\sin\theta)\,dr\,d\theta\,d\phi$$

Hence ϕ varies from 0 to 2π

ϕ varies from 0 to π

ϕ varies from 0 to 1

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \frac{1}{\sqrt{1-r^2}}\, r^2\sin\theta\, dr\,d\theta\,d\phi$$

$$= \left[\int_0^{2\pi} d\phi\right]\left[\int_0^{\pi}\sin\theta\,d\theta\right]\left[\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\,dr\right]$$

$$= [\phi]_0^{2\pi}\,[-\cos\theta]_0^{\pi}\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\,dr$$

$$= (2\pi - 0)\,(1+1)\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\,dr$$

$$= 4\pi\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\,dr$$

Put $r = \sin t$; $dr = \cos t\,dt$

$r = 0 \Rightarrow t = 0$

$r = 1 \Rightarrow t = \frac{\pi}{2}$

$$= 4\pi\int_0^{\pi/2} \frac{\sin^2 t}{\sqrt{1-\sin^2 t}}\cos t\,dt$$

$$= 4\pi\int_0^{\pi/2} \frac{\sin^2 t}{\sqrt{\cos^2 t}}\cos t\,dt$$

$$= 4\pi\int_0^{\pi/2} \frac{\sin^2 t}{\cos t}\cos t\,dt$$

$$= 4\pi\int_0^{\pi/2} \sin^2 t\,dt$$

$$= 4\pi\,\frac{1}{2}\,\frac{\pi}{2} = \pi^2$$

Example:

Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^1 \frac{dz\,dy\,dx}{\sqrt{x^2+y^2+z^2}}$

Solution:

Given x varies from 0 to 1

y varies from 0 to $\sqrt{1-x^2}$

z varies from $\sqrt{x^2+y^2}$ to 1

Let us transform this integral into spherical polar co – ordinates by using

$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$dx\,dy\,dz = (r^2\sin\theta)\,dr\,d\theta\,d\phi$$

Let $z = \sqrt{x^2+y^2}$

$\Rightarrow z^2 = x^2 + y^2$

$\Rightarrow r^2\cos^2\theta = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi$

$\Rightarrow \cos^2\theta = \sin^2\theta$ $\quad [\because \cos^2\phi + \sin^2\phi = 1]$

$\Rightarrow \theta = \frac{\pi}{4}$

Let $z = 1$

$\Rightarrow r\cos\theta = 1$

$\Rightarrow r = \frac{1}{\cos\theta}$

$\Rightarrow r = \sec\theta$

The region of integration is common to the cone $z^2 = x^2 + y^2$ and the cylinder $x^2 + y^2 = 1$ bounded by the plane $z = 1$ in the positive octant.

Limits of r : $r = 0$ to $r = \sec\theta$

Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{4}$

Limits of ϕ : $\phi = 0$ to $\phi = \frac{\pi}{2}$

$$= \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\theta}\frac{1}{r}\,r^2\sin\theta\,dr\,d\theta\,d\phi \qquad = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\theta} r\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/4}\left[\sin\theta\,\frac{r^2}{2}\right]_0^{\sec\theta} d\theta\,d\phi \qquad = \int_0^{\pi/2}\int_0^{\pi/4}\left[\frac{\sec^2\theta\sin\theta - 0}{2}\right] d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/4}\frac{1}{2}\sec\theta\tan\theta\,d\theta\,d\phi \qquad = \left[\frac{1}{2}\int_0^{\pi/2} d\phi\right]\left[\int_0^{\pi/4}\sec\theta\tan\theta\,d\theta\right]$$

$$= \frac{1}{2}[\,\theta\,]_0^{\pi/2}\,[\sec\theta]_0^{\pi/4}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - 0\right]\left[\sqrt{2} - 1\right]$$

$$= \frac{\pi}{4}\left(\sqrt{2} - 1\right)$$

Example:

Evaluate $\int\int\int (x^2 + y^2 + z^2)\,dx\,dy\,dz$ taken over the region bounded by the volume enclosed by the sphere $x^2 + y^2 + z^2 = 1$.

Solution:

Let us convert the given integral into spherical polar co – ordinates.

$$x = r\sin\theta\cos\phi \quad \Rightarrow x^2 = r^2\sin^2\theta\cos^2\phi$$

$$y = r\sin\theta\sin\phi \quad \Rightarrow y^2 = r^2\sin^2\theta\sin^2\phi$$

$$z = r\cos\theta \quad \Rightarrow z^2 = r^2\cos^2\theta$$

$$dx\,dy\,dz = (r^2\sin\theta)\,dr\,d\theta\,d\phi$$

$$\int\int\int (x^2 + y^2 + z^2)\,dx\,dy\,dz = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} r^2\,(r^2\sin\theta\,d\theta\,d\phi\,dr)$$

Limits of r : $r = 0$ to $r = 1$

Limits of θ : $\theta = 0$ to $\theta = \pi$

Limits of ϕ : $\phi = 0$ to $\phi = 2\pi$

$$\int\int\int (x^2 + y^2 + z^2)\,dx\,dy\,dz = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} r^2\,(r^2\sin\theta\,d\theta\,d\phi\,dr)$$

$$= \left[\int_0^1 r^4\,dr\right]\left[\int_0^{\pi}\sin\theta\,d\theta\right]\left[\int_0^{2\pi} d\phi\right]$$

$$= \left[\frac{r^5}{5}\right]_0^1 [-\cos\theta]_0^{\pi}\,[\phi]_0^{2\pi}$$

$$= \left(\frac{1}{5} - 0\right)(1 + 1)(2\pi - 0)$$

$$= \left(\frac{1}{5}\right)(2)(2\pi) = \frac{4\pi}{5}$$